Linear Model of Population Dynamics

Malthus (1798)

Assumption

Population $P(t)$ at time t
grows with a constant rate r.

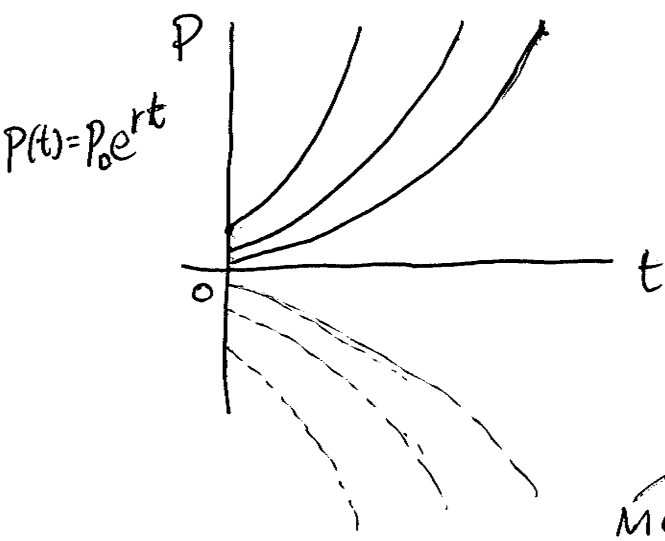
$$\boxed{P' = rP} \quad \text{(Linear Eq.)}$$

where r = the net per capita growth rate (say, 2.5%)
$= b - d$ (birth rate − death rate)

Assume r is constant (invariant with time).

I.V.P.: $\begin{cases} P' = rP \\ P(0) = P_0 \end{cases}$ $\Rightarrow$ Sol. $P(t) = P_0 e^{rt}$.

When $r > 0$, population grows.

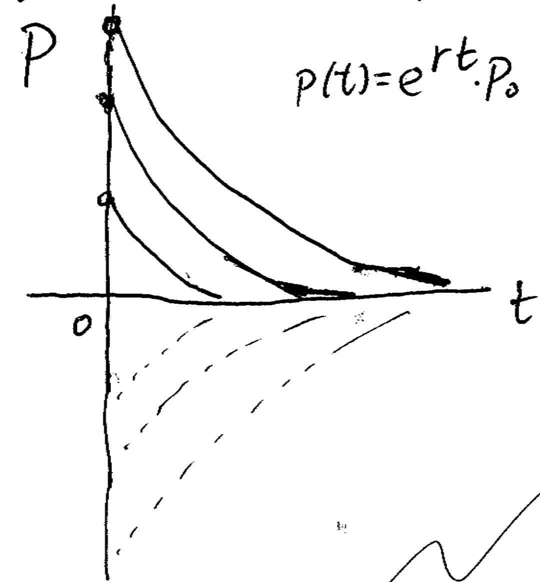


- Positive Sol's. $P(t) \nearrow \infty$ as $t \to \infty$.
- $P = 0$ is an equilibrium Sol.
- Equilibrium $P = 0$ is "unstable".

Meaning of "instability":

Although initial Cond $P_0 = 0 \Rightarrow P(t) \equiv 0$,
a small initial Cond $P_0 \approx 0 \Rightarrow$ A Sol. $P(t)$ does NOT remain small as $t \to \infty$.

When $r<0$, population decays.



$p(t)=e^{rt}\cdot P_0$

- All solutions $P(t)\to 0$ as $t\to\infty$
- $P=0$ is an equilibrium
- Equilibrium $P=0$ is "stable"
- Equilibrium $P=0$ is "asymptotically stable".

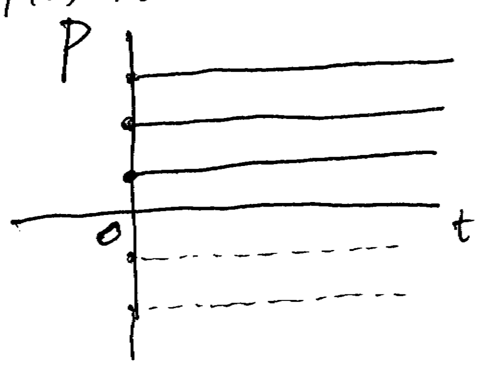
"Stability" means a small initial cond $P_0\approx 0$ gives a sol. $P(t)$ that remains small

"Asymptotical Stability" means

$$\begin{cases}\bullet\ \text{"stable": When } P_0\approx 0,\ P(t)\approx 0 \text{ for } t>0.\\ \bullet\ \text{Moreover, when } P_0\approx 0,\ \lim_{t\to\infty}P(t)=0.\end{cases}$$

When $r=0$, population does not change.

$$\begin{cases}P'=0\\ P(0)=P_0\end{cases}\Rightarrow P(t)\equiv P_0$$



- All Sol's $P(t)$ are equilibria
- Each equilibrium is "stable", but not "asymp. stable".

Stability, Asymptotic Stability, & Instability

$$\boxed{y' = f(y)}$$

Let $y(t) \equiv a$ be an equilibrium sol. i.e. $f(a) = 0$.

- Equilibrium a is "stable"
  if [$y(t)$ remains close to a for all $t > 0$ as long as $y(0)$ is close enough to a.

- Equilibrium a is "asymptotically stable"
  if [as long as $y(0)$ is close enough to a, $y(t)$ remains close to a for all $t > 0$, & $\lim_{t \to \infty} y(t) = a$

- Equilibrium a is "unstable"
  if a is not stable,
  that is, there are some $y(0) \approx a$, that give $y(t)$ leaving the vicinity of a.

Logistic Model of Population Dynamics

Verhulst (1838)

$\boxed{\text{Assumption}}$

- Growth Rate decreases as population increases
- The Limit of Resource implies a finite "Carrying capacity".

$$\boxed{P' = r\left(1 - \frac{P}{K}\right) P} \qquad \text{(Nonlinear Eq)}$$

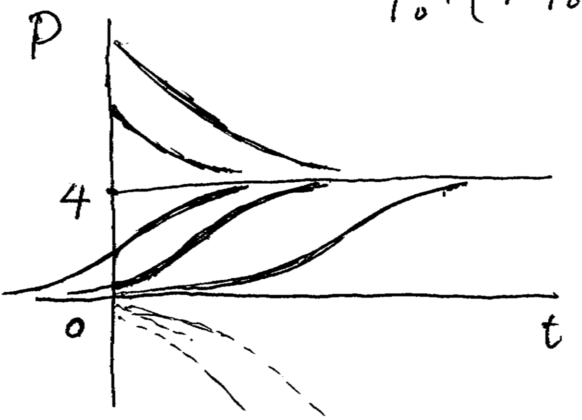
where

- r = the growth rate when P is small (i.e. when the resource limit is unimportant)
- K = the "carrying capacity".

Sol. Formula $\quad P(t) = \dfrac{K P_0}{P_0 + (K - P_0) e^{-rt}}$

Example $\quad P' = 0.03\left(1 - \frac{P}{4}\right) P. \qquad (r = 0.03,\ K = 4.)$

Sol. $\quad P(t) = \dfrac{4 P_0}{P_0 + (4 - P_0) e^{-0.03t}}$



Solution Graphs.

- $P = 0$ is an equilibrium
- $P = 4$ is an equilibrium — $P = K$.
- All positive Sol's. $P(t) \to 4$ as $t \to \infty$.
- $P = 0$ is an "unstable" equilibrium
- $P = 4$ is an "asymp. stable" equilibrium