Population Growth / Decay

Linear Exp. Model (Malthus, 1798)

Assumption Population y changes exponentlially.

i.e. $\frac{dy}{dt} = ky$. $y(t) = y_0 e^{kt}$.

Here, k = the net rate of change per unit of populatio

$= \begin{pmatrix}\text{the birth rate} \\ \text{per capita}\end{pmatrix} - \begin{pmatrix}\text{the death rate} \\ \text{per capita}\end{pmatrix}$.

Example A culture of bacteria grows exponentially.

Observed: # of bacteria increased 25 % in a hour.

Question: How long does it take for the # of bacteria to double?

Solution: $y(t) = y_0 e^{kt}$.

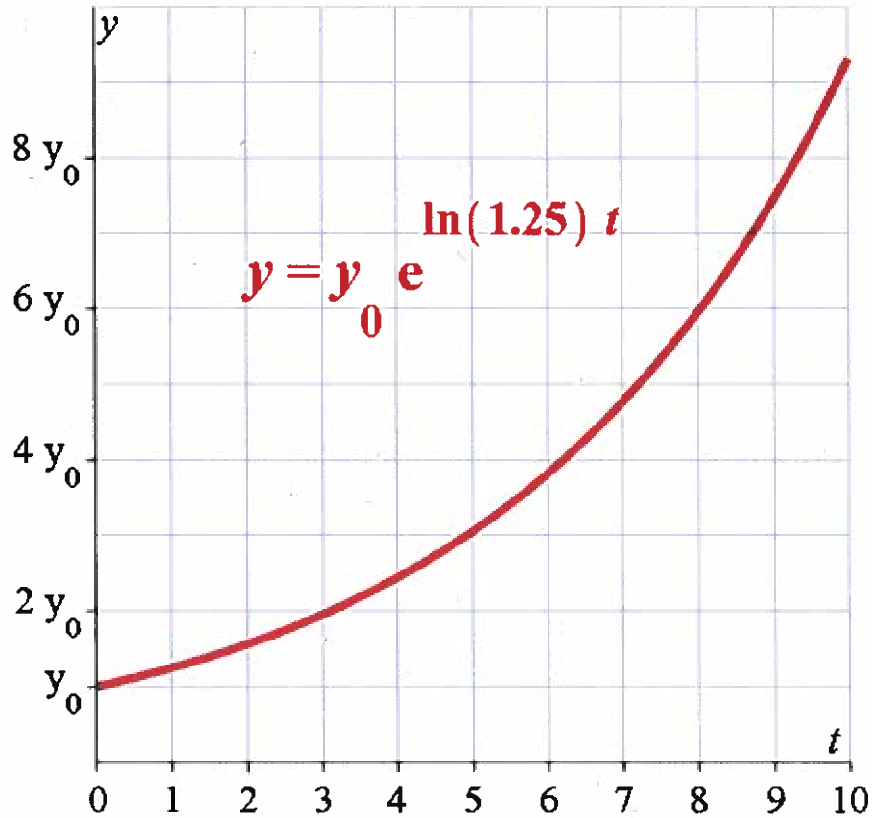
$y(1) = 1.25 y_0 = y_0 e^{k \cdot 1} \Rightarrow 1.25 = e^{k} \Rightarrow k = \ln 1.25$

$y(t) = y_0 e^{(\ln 1.25)t}$.

Q. Find t such that $y(t) = 2y_0$.

$2y_0 = y_0 e^{(\ln 1.25)t} \Rightarrow 2 = e^{(\ln 1.25)t}$

$\Rightarrow \ln 2 = (\ln 1.25)t \Rightarrow t = \frac{\ln 2}{\ln 1.25}$ hours



Answer : Time to Double : $t_2 = \dfrac{\ln 2}{\ln 1.25} \approx 3.1$ hours

By the way: Time to Triple : $t_3 = \dfrac{\ln 3}{\ln 1.25} \approx 4.9$ hours

Time to Quadruple: $t_4 = \dfrac{\ln 4}{\ln 1.25} \approx 6.2$ hours

Time to Octuple : $t_8 = \dfrac{\ln 8}{\ln 1.25} \approx 9.3$ hours

we always have $t_4 = 2t_2$,

$t_8 = 3t_2$.

Q. Why?

Example

- Coffee Temp. $T(t)$
- Room Temp. 70°F (Surrounding Temp.)
  constant

Newton's Law of Cooling

The rate of decrease of Coffee Temp.
$\propto$ Coffee Temp. − Room Temp.

$$-\frac{dT}{dt} = k(T-70), \qquad T(0) = 190°F.$$

[An initial value problem of diff. eq.]

Solution:

Set $y = T - 70$.

$$-\frac{dy}{dt} = ky. \qquad \begin{cases} \frac{dy}{dt} = -ky \\ y(0) = 190 - 70 = 120. \end{cases}$$

$\swarrow$

Solution: $y(t) = 120e^{-kt}$

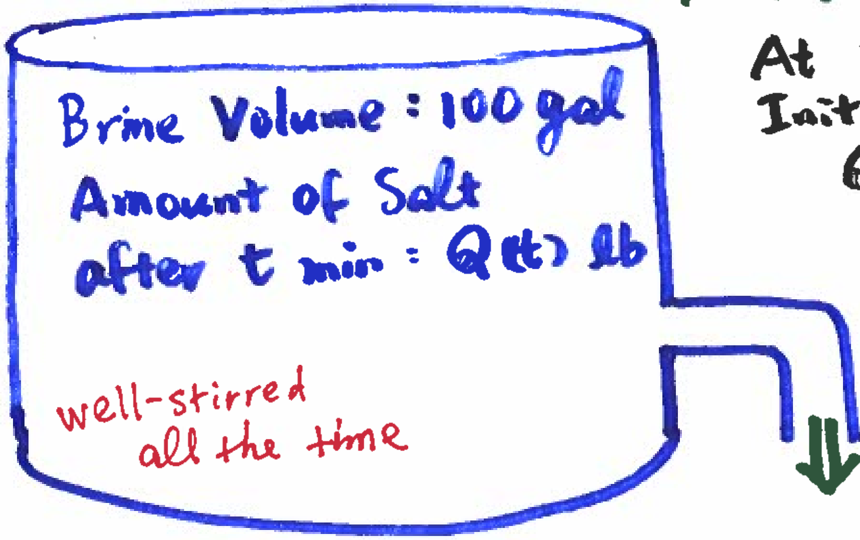
Coffee Temp. $T(t) = 70 + y$

$$= 70 + 120e^{-kt}$$

Salt in a Tank

Rate of Brine In : $r = 3$ gal/min

Salt Concentration: $\frac{1}{4}$ lb/gal



At $t=0$:
Initial Amount of Salt :
$Q(0) = 10$ lb

Rate of Brine Out :
$r = 3$ gal/min

Question Find $Q(t)$, the amount of Salt after t min.

Set up a diff eq for $Q(t)$:

(rate of change of Salt) = (rate of Salt in) − (rate of Salt out)

$$\frac{dQ}{dt} = (3)\left(\frac{1}{4}\right) - (3)\left(\frac{Q}{100}\right)$$

$$\left.\begin{array}{l} \text{Diff Eq : } \dfrac{dQ}{dt} = \dfrac{3}{4} - \dfrac{3}{100}Q \\ \text{Init. Cond: } Q(0) = 10 \end{array}\right\} \text{ Initial Value Problem}$$

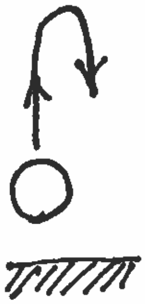
Solution Method 1 : Separate the Variables Q and t.

Solution Method 2 : Integrating Factor

Solution Method 3 : Set $y = Q - 25$.

$$\left\{\begin{array}{l} \dfrac{dy}{dt} = -\dfrac{3}{100}y \\ y(0) = -15 \end{array}\right. \Rightarrow y = Ce^{-\frac{3}{100}t} = -15e^{-\frac{3}{100}t} \Rightarrow Q = 25 + y = 25 - 15e^{-\frac{3}{100}t} \text{ (lb)}.$$

A Ball Thrown Up & Falling Down



Newton's 2nd Law of Mechanics:

(Mass)(Acceleration) = (Net Force on the Mass)

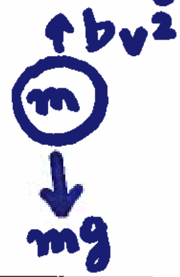
- Assume Forces = $\begin{cases} \text{Gravity} & mg \\ \text{Air Resistance} & bv^2 \end{cases}$
- Let Upward Velocity > 0

Rising Up:

$$m\frac{dv}{dt} = -mg - bv^2$$

$$\frac{dv}{dt} = -g - \frac{b}{m}v^2$$

Falling Down:



$$m\frac{dv}{dt} = -mg + bv^2$$

$$\frac{dv}{dt} = -g + \frac{b}{m}v^2$$

Terminal Speed of Free Fall, V_T:

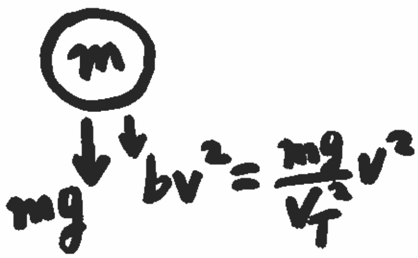
$$-g + \frac{b}{m}V_T^2 = 0 \Rightarrow V_T = \sqrt{\frac{mg}{b}}, \text{ or } \frac{b}{m} = \frac{g}{V_T^2}.$$

$$\boxed{\begin{aligned} \text{Up:} \quad & \frac{dv}{dt} = -g - \frac{g}{V_T^2}v^2 \\ \text{Down:} \quad & \frac{dv}{dt} = -g + \frac{g}{V_T^2}v^2 \end{aligned}}$$

Objects	Terminal Speed
Basketball	20 m/s
Baseball	40 m/s
Ping Pong	9 m/s
Skydiver (parachute)	5–9 m/s
Skydiver (spread-eagle)	50–60 m/s
Skydiver (dive)	100 m/s
Raindrop	7 m/s

Solve the diff eq:

Rising Up:



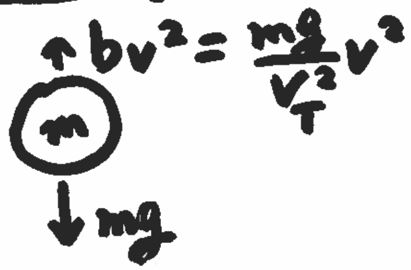
$$\frac{dv}{dt} = -g - \frac{g}{V_T^2}v^2$$

Separable Diff Eq

$$\int \frac{dv}{1+\frac{v^2}{V_T^2}} = \int -g\,dt, \quad V_T \arctan\frac{v}{V_T} = -gt + C,$$

$$v = V_T \tan\left(\frac{-gt+C}{V_T}\right)$$

Falling Down:



$$\frac{dv}{dt} = -g + \frac{g}{V_T^2}v^2$$

$$= \frac{g}{V_T^2}(v^2 - V_T^2)$$

$$\int \frac{dv}{v^2 - V_T^2} = \int \frac{g}{V_T^2}dt,$$

$$\frac{1}{V^2 - V_T^2} = \frac{1}{2V_T}\left(\frac{1}{V-V_T} - \frac{1}{V+V_T}\right)$$

Partial Fractions

$$\frac{1}{2V_T}\left(\ln|v-V_T| - \ln|v+V_T|\right) = \frac{g}{V_T^2}t + C_1.$$

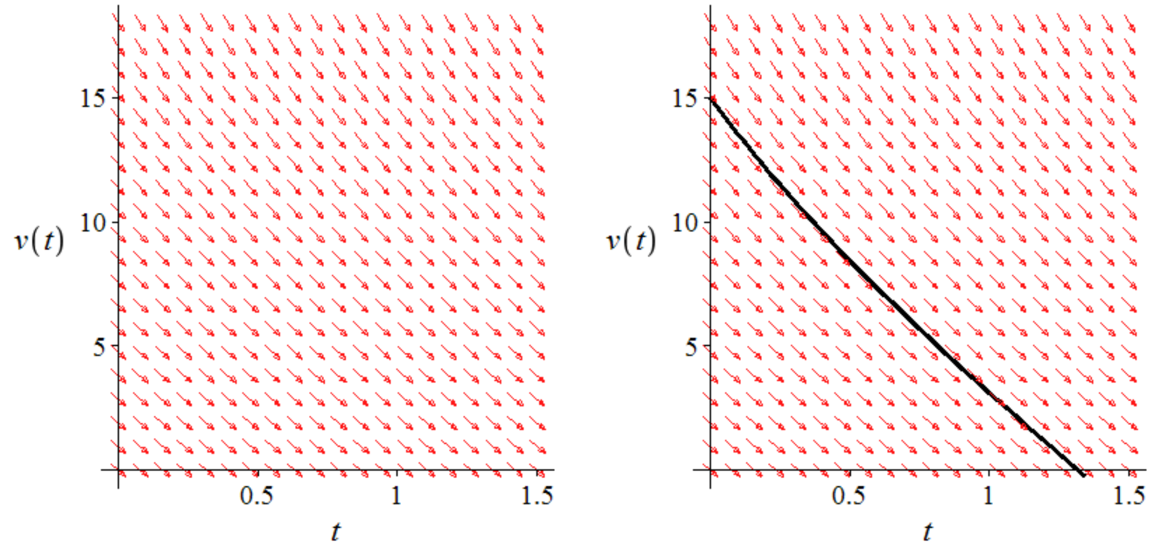
$$= \frac{1}{2V_T}\ln\left|\frac{v-V_T}{v+V_T}\right|, \quad \ln\left|\frac{v-V_T}{v+V_T}\right| = \frac{2g}{V_T}t + 2V_T C_1$$

$$\frac{v-V_T}{v+V_T} = \pm e^{\frac{2g}{V_T}t + 2V_T C_1} = Ce^{\frac{2g}{V_T}t} \Rightarrow v = V_T\frac{1+Ce^{\frac{2g}{V_T}t}}{1-Ce^{\frac{2g}{V_T}t}}$$

Gravitational acceleration: $g = 9.8\ m/s^2$
Basketball terminal speed: $v_T = 20\ m/s$

Rising:

$$\frac{dv}{dt} = g\left(-1 - \frac{v^2}{v_T^2}\right)$$



Falling:

$$\frac{dv}{dt} = g\left(-1 + \frac{v^2}{v_T^2}\right)$$