Exponential Growth/Decay

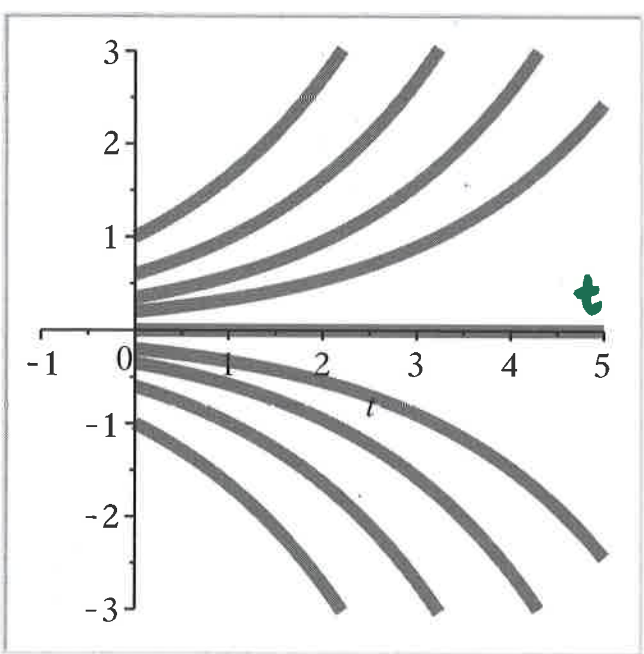
Often ① A quantity changes with time
② The rate of change is proportional to the present size of the quantity.

Equation ① $y = y(t)$ ② $\frac{dy}{dt} = ky$

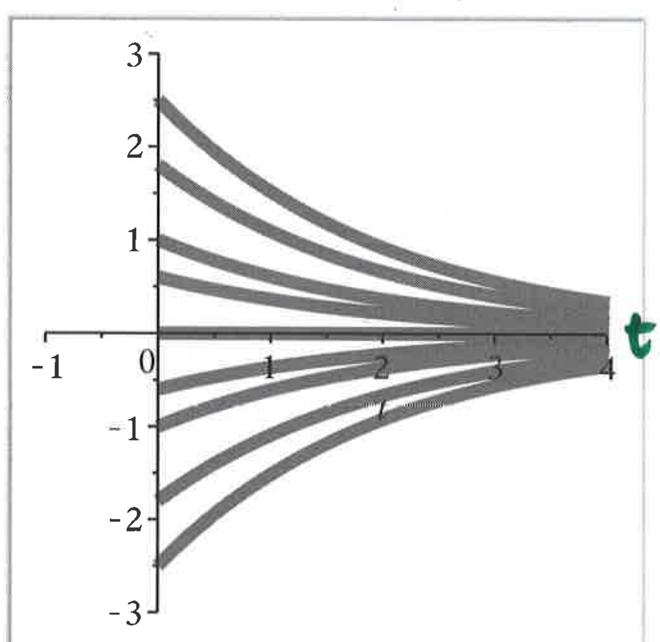
Initial Value Problem

$\frac{dy}{dt} = ky$, $y(0) = y_0$ $\Rightarrow$ Solution $y(t) = y_0 e^{kt}$

$k>0$: Exp. Growth



$k<0$: Exp. Decay



k measures how fast the growth/decay is.

How to Solve $\frac{dy}{dt} = ky$?

Method 1 (Integrating Factor)

$$\frac{dy}{dt} - ky = 0 .$$ multiply both sides by e^{-kt}

$$\Rightarrow e^{-kt}\frac{dy}{dt} - ke^{-kt}y = 0,$$

$$\Rightarrow \frac{d}{dt}(e^{-kt}y) = 0,$$

$$\Rightarrow e^{-kt}y = C \Rightarrow \boxed{y = Ce^{kt}}$$

Method 2 (Separation of Variables).

$$\frac{dy}{dt} = ky \Rightarrow \frac{1}{y}dy = k\,dt$$

$$\Rightarrow \int \frac{1}{y}dy = \int k\,dt, \quad \ln|y| = kt + C_1$$

$$\Rightarrow |y| = e^{kt+C_1} = e^{C_1}\cdot e^{kt}$$

$$\Rightarrow y = \pm e^{C_1}\cdot e^{kt}$$

$$\boxed{y = Ce^{kt}}$$

Population Growth / Decay

Linear Exp. Model (Malthus, 1798)

<u>Assumption</u> Population y changes exponentially.

i.e. $\frac{dy}{dt} = ky$. $\quad y(t) = y_0 e^{kt}$.

Here, k = the net rate of change per unit of population

$= \begin{pmatrix}\text{the birth rate} \\ \text{per capita}\end{pmatrix} - \begin{pmatrix}\text{the death rate} \\ \text{per capita}\end{pmatrix}$.

<u>Example</u> A culture of bacteria grows exponentially.

<u>Observed</u>: # of bacteria increased 25 % in a hour.

<u>Question</u>: How long does it take for the # of bacteria to double?

<u>Solution</u>: $y(t) = y_0 e^{kt}$.

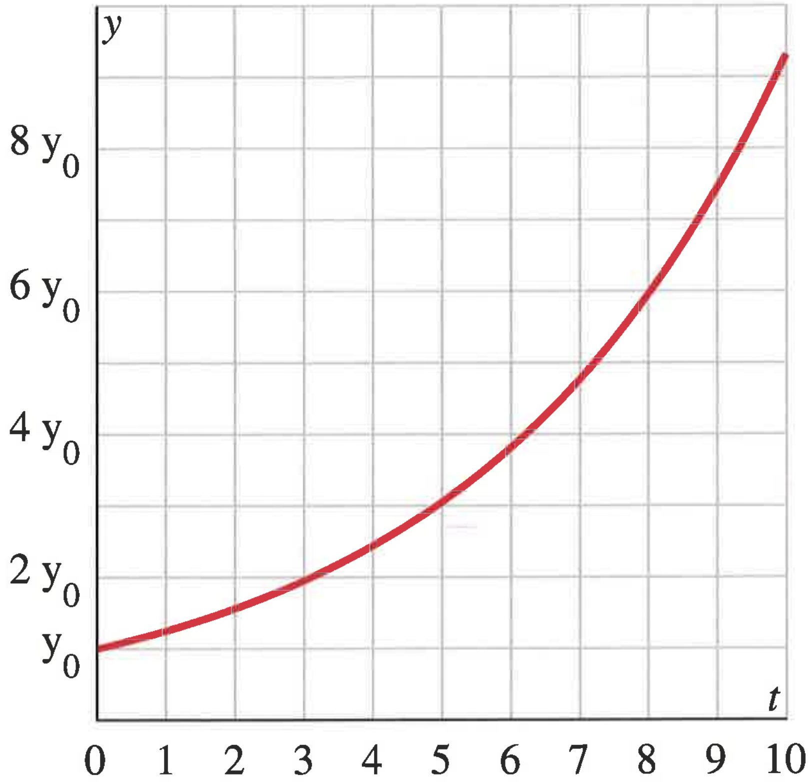
$y(1) = 1.25 y_0 = y_0 e^{k \cdot 1} \Rightarrow 1.25 = e^{k} \Rightarrow k = \ln 1.25$

$y(t) = y_0 e^{(\ln 1.25)t}$.

<u>Q.</u> Find t such that $y(t) = 2y_0$.

$2y_0 = y_0 e^{(\ln 1.25)t} \Rightarrow 2 = e^{(\ln 1.25)t}$

$\Rightarrow \ln 2 = (\ln 1.25)t \Rightarrow t = \frac{\ln 2}{\ln 1.25}$ hours



$$y = y_0 e^{(\ln 1.25)\, t}$$

Answer: Time to Double : $t_2 = \frac{\ln 2}{\ln 1.25} \approx 3.1$ hours

By the way:

Time to Triple : $t_3 = \frac{\ln 3}{\ln 1.25} \approx 4.9$ hours

Time to Quadruple: $t_4 = \frac{\ln 4}{\ln 1.25} \approx 6.2$ hours

Time to Octuple: $t_8 = \frac{\ln 8}{\ln 1.25} \approx 9.3$ hours

we always have $t_4 = 2t_2$,
$t_8 = 3t_2$.

Q. Why?

Example

- Coffee Temp. $T(t)$
- Room Temp. $70°F$ (Surrounding Temp.)
  constant

Coffee

Newton's Law of Cooling

The rate of decrease of Coffee Temp.
$\propto$ Coffee Temp. − Room Temp.

$$-\frac{dT}{dt} = k(T-70), \qquad T(0) = 190°F.$$

[An initial value problem of diff. eq.]

Solution:

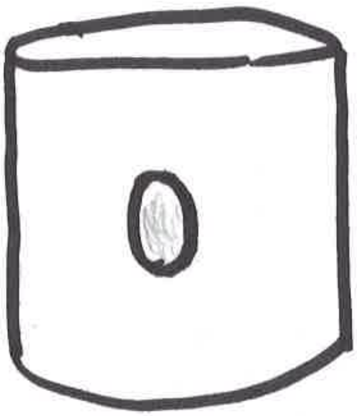
Set $y = T - 70$.

$$-\frac{dy}{dt} = ky. \qquad \begin{cases} \dfrac{dy}{dt} = -ky \\ y(0) = 190 - 70 = 120. \end{cases}$$

Solution: $y(t) = 120e^{-kt}$

Coffee Temp. $T(t) = 70 + y$
$= 70 + 120e^{-kt}$

Example A Hot Egg in Water.



Egg Temp. $= T$: changes with t

Water Temp. $=$ Constant $= T_s$
(Surrounding Temp.)

Assume : Initial Egg Temp. $T_0 = 98^\circ C$ $(t=0)$
Constant Water Temp. $T_s = 18^\circ C$.
Egg Temp. after 5 min: $T(5) = 38^\circ C$.

Q. Find t such that $T(t) = 20^\circ C$.

Solution Newton's Cooling Law: $\frac{dT}{dt} = -k(T - T_s)$

Set: $y = T - T_s \Rightarrow \frac{dy}{dt} = -ky$

$\Rightarrow y = y_0 e^{-kt}$

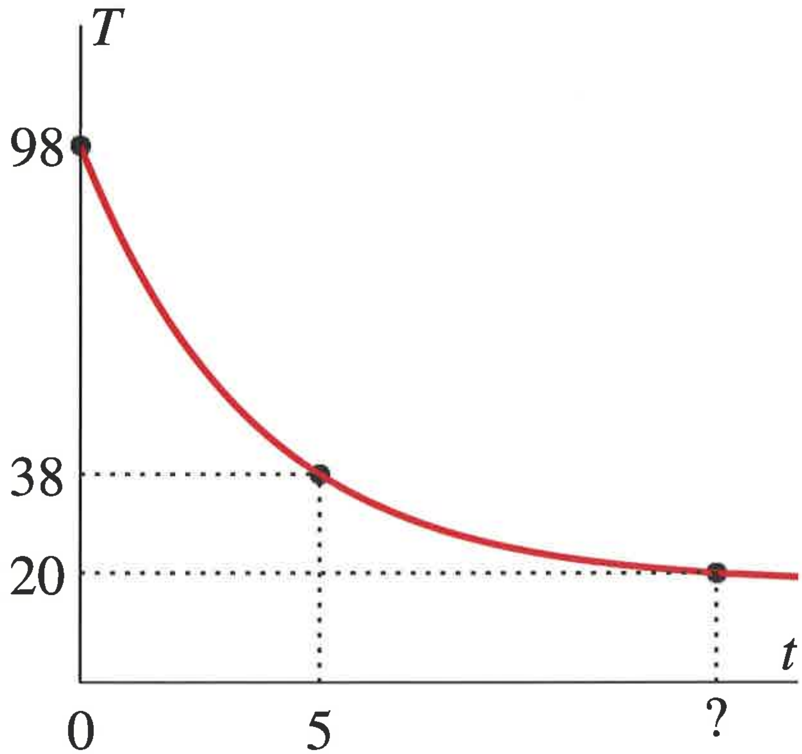
$T - T_s = (T_0 - T_s)e^{-kt}$, $T - 18 = (98-18)e^{-kt}$

$T(t) = 18 + 80e^{-kt}$.

$\boxed{t=5}$: $38 - 18 = 80e^{-5k} \Rightarrow e^{-5k} = \frac{20}{80} = \frac{1}{4}$,

$-5k = \ln\frac{1}{4} = -\ln 4$, $k = \frac{1}{5}\ln 4$.

Eq: $20 - 18 = 80e^{-(\frac{1}{5}\ln 4)t}$, $-(\frac{1}{5}\ln 4)t = \underbrace{\ln\frac{2}{80}}_{= -\ln 40} \Rightarrow t = \frac{5\ln 40}{\ln 4}$



$$T(t) = 18 + 80\,e^{-kt}$$

$$T(0) = 98,\ T(5) = 38 \Rightarrow \boldsymbol{k} = \frac{1}{5}\ln 4$$

$$T(t) = 18 + 80\,e^{-\left(\frac{1}{5}\ln 4\right)t},\quad \boldsymbol{T}(\boldsymbol{t}) = 25 \Rightarrow t = ?$$

Ans: $t = \dfrac{5\ln 40}{\ln 4}$ min ≈ 13.30 min

$\approx$ 13 min 18 sec.

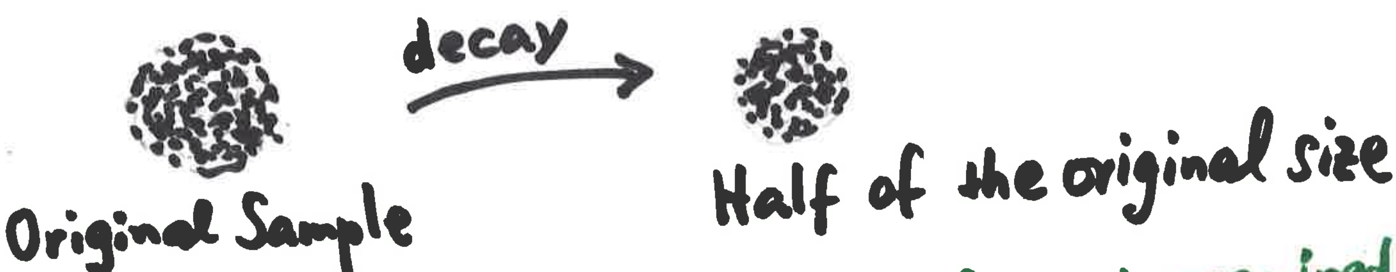
Radioactive Decay

Physics $\Rightarrow$ Radioactive Matters Decay Exponentially.

y = the amount of radioactive

$$\frac{dy}{dt} = -ky \quad (k>0 : \text{constant}) \qquad y(t) = y_0 e^{-kt}$$

- k : the decay rate (measures how fast the decay is).
- Half-Life : also measures how fast the decay is.



Half-Life h = the time length required for decaying into half.

$$y(h) = y_0 e^{-kh} = \frac{1}{2} y_0 \Rightarrow e^{-kh} = \frac{1}{2}$$

$$-kh = \ln\frac{1}{2} = -\ln 2$$

$$\boxed{kh = \ln 2}$$

Carbon Dating

C-12 $\left\{\begin{matrix} 6 \text{ neutrons} \\ 6 \text{ protons} \end{matrix}\right)$: stable carbon

C-14 $\left(\begin{matrix} 8 \text{ neutrons} \\ 6 \text{ protons} \end{matrix}\right)$: radioactive isotope

Facts from Physics

- In a living organism :

$$\frac{\text{Amount of C-14}}{\text{Amount of C-12}} = \frac{\text{C-14 in Atmosphere}}{\text{C-12 in Atmosphere}} = \text{Const.}$$

$$= 10^{-12}$$

- When an organism dies :

$$\begin{cases} \text{C-14 : decays exponentially with } \underline{\text{half-life}} \text{ 5730 years} \\ \text{C-12 : NO change} \end{cases}$$

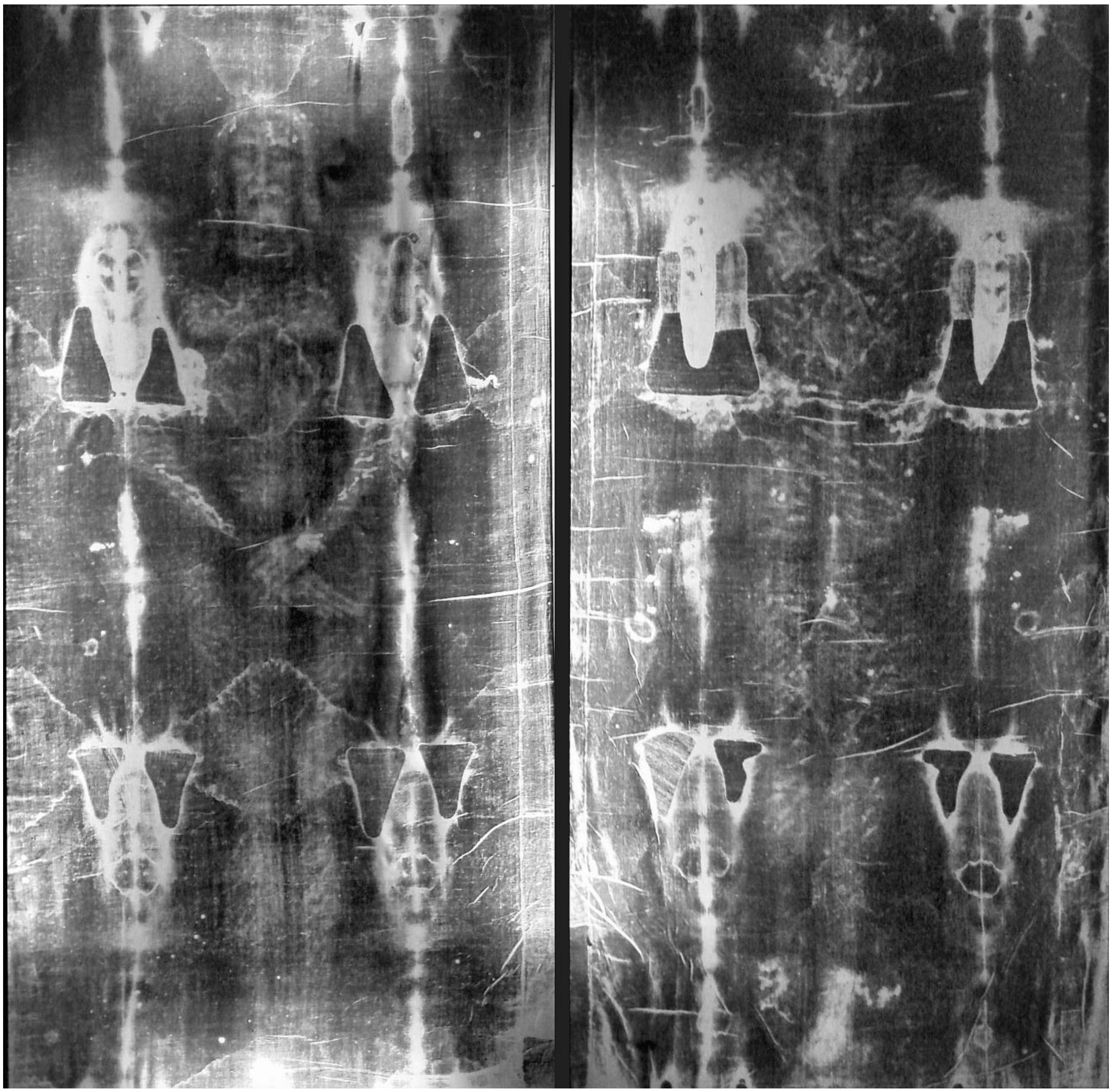
Method of Carbon Dating

In an ancient (prehistoric) artifact, measure the levels of C-14 & C-12.

$\Rightarrow$ the age of the artifact

Example. **The Shroud of Turin.**

Some people believed that it was used to bury Jesus Christ.



Reference: http://en.wikipedia.org/wiki/Shroud_of_Turin

Example (Shroud of Turin)

Tested in 1988:

(Amount of C-14) = 92.27% of the original size

$\Rightarrow$ Age of Shroud = ?

Solution (half-life of C-14) = 5730

Amount of C-14: $y(t) = y_0 e^{-kt}$

$$k \cdot 5730 = \ln 2, \quad k = \frac{\ln 2}{5730}.$$

$$y(t) = y_0 e^{-\left(\frac{\ln 2}{5730}\right)t}$$

$$y_0 e^{-\left(\frac{\ln 2}{5730}\right)t} = 0.9227\, y_0$$

$$\Rightarrow -\left(\frac{\ln 2}{5730}\right)t = \ln 0.9227$$

age

$$t = \frac{-\ln 0.9227}{\ln 2} \cdot 5730 \approx 665 \text{ years}$$

$\Rightarrow 1988 - 665 = 1323$.

Conclusion: The shroud was made around the year of 1323.