Phase Portrait Method for Autonomous Diff Eq $\frac{dy}{dt} = f(y)$

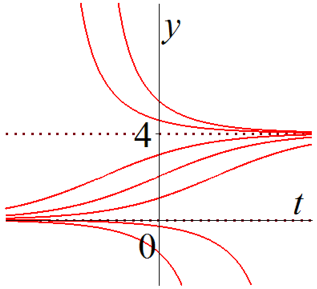
$$\frac{dy}{dt} = 0.03\left(1 - \frac{y}{4}\right)y$$

Compare the three different methods of determining the behavior of solutions.

Method 1: Solve the Diff Eq, Get the Solution Formula, Sketch Solution Graphs vs t, and Determine the Dynamic Behavior of Solutions.

Solve the diff eq to get the solution formula:

$$y(t) = \frac{4y_0}{y_0 + (4 - y_0)e^{-0.03t}}$$

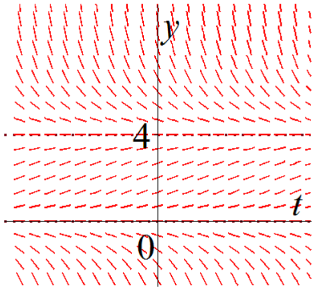


Equilibria: $y = 0, y = 4.$

$y = 0$ is unstable.

$y = 4$ is asymptotically stable.

Method 2: Sketch the Direction Field (Slope Field), and Determine the Dynamic Behavior of Solutions.



Equilibria: $y = 0, y = 4.$

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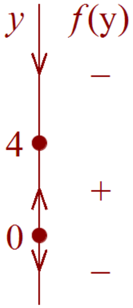
Method 3: Sketch the Phase Portrait, and Determine the Dynamic Behavior of Solutions.



$$f(y) = 0.03\left(1 - \frac{y}{4}\right)y = 0$$

$\Rightarrow$ Equilibria: $y = 0, y = 4.$

$y = 0$ is unstable.

$y = 4$ is asympt. stable.

Phase Portraits

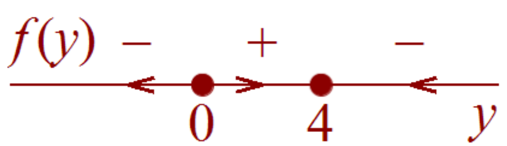
Determine the solution behavior without solving the diff eq

Example: $\frac{dy}{dt} = 0.03\left(1 - \frac{y}{4}\right)y.$ $\qquad f(y) = 0.03\left(1 - \frac{y}{4}\right)y$

- Equilibria (i.e., time-independent solutions, constant sol's)

  $$\Leftrightarrow \frac{dy}{dt} = 0 \Leftrightarrow f(y) = 0 \Leftrightarrow 0.03\left(1 - \frac{y}{4}\right)y = 0 \Leftrightarrow y = 0, y = 4.$$

- Increasing Solutions $\Leftrightarrow \frac{dy}{dt} > 0 \Leftrightarrow f(y) > 0$
- Decreasing Solutions $\Leftrightarrow \frac{dy}{dt} < 0 \Leftrightarrow f(y) < 0$
- **Phase Portrait**



Phase Portraits

Determine the solution behavior without solving the diff eq

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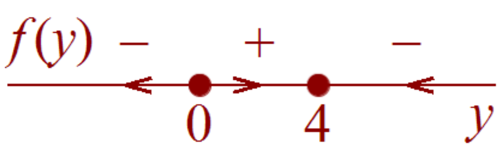

- $y = 0$ is unstable.
  $y = 4$ is asymptotically stable.

For $y(0)\approx 0$, do we always have $y(t)\approx 0$ for all $t>0$?

For $y(0)\approx 4$, do we always have

- $y(t)\approx 4$ for all $t>0$; and
- $\lim_{t\to\infty} y(t) = 4$?

Phase Portraits

Determine the solution behavior without solving the diff eq

Example: $\frac{dy}{dt} = 0.03\left(1 - \frac{y}{4}\right)y.$ $\qquad f(y) = 0.03\left(1 - \frac{y}{4}\right)y$

- Equilibria (i.e., time-independent solutions, constant sol's)

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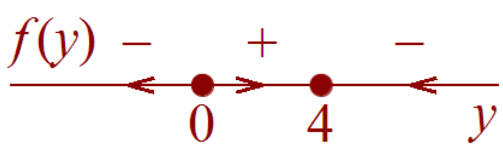
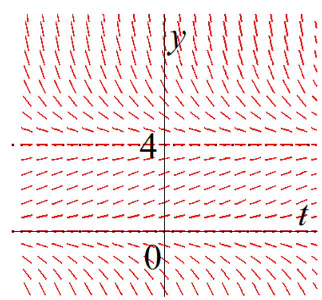
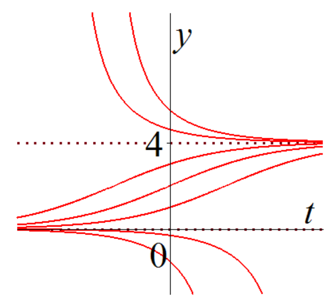


Direction Field



Solution Graphs



- $y = 0$ is unstable.
  $y = 4$ is asymptotically stable.

Phase Portrait Method for Autonomous Diff Eq $\frac{dy}{dt} = f(y)$

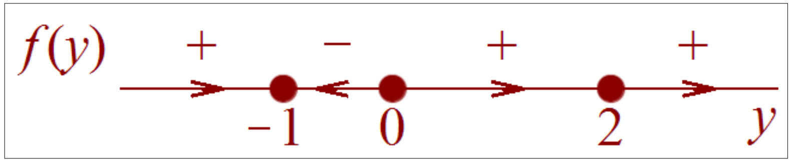
- Find equilibria $\Leftrightarrow \frac{dy}{dt} = f(y) = 0$.
- On the intervals between the equilibria, determine the signs of $f(y)$.
- Sketch the phase portrait.
- Phase portrait gives
  - stability, asymptotic stability, or instability;
  - approximate direction field;
  - approximate solution graphs y vs t.

Example: $\frac{dy}{dt} = (e^y - 1)(y-2)^2(y+1)$.

(a) Find all equilibria.
(b) Sketch the phase portrait.
(c) Determine whether each equilibrium is stable, asymptotically stable, or unstable.

a) Equilibria $\Leftrightarrow \frac{dy}{dt} = 0 \Leftrightarrow f(y) = 0 \Leftrightarrow y = 0, y = 2, y = -1.$

b) Phase Portrait



c) $y = -1$ is asymptotically stable.
$y = 0$ is unstable.
$y = 2$ is unstable.

Remark: This type of $y = 2$ is also said to be semi-stable.

Example: $\frac{dy}{dt} = (e^y - 1)(y-2)^2(y+1).$

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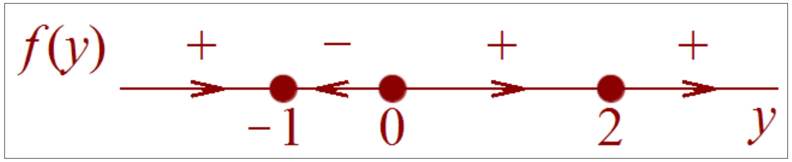

c) $y = -1$ is asymptotically stable.
$y = 0$ is unstable.
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For $y(0) \approx -1$, do we always have
- $y(t) \approx -1$ for all $t>0$; and
- $\lim_{t\to\infty} y(t) = -1$?

For $y(0) \approx 0$, do we always have $y(t) \approx 0$ for all $t>0$?

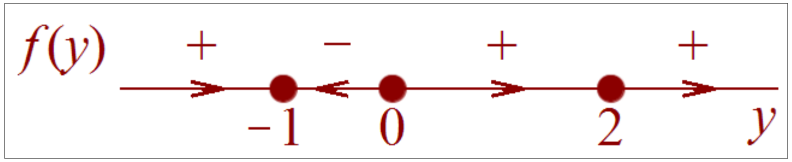
For $y(0) \approx 2$, do we always have $y(t) \approx 2$ for all $t>0$?

Example: $\frac{dy}{dt} = (e^y - 1)(y-2)^2(y+1).$

(a) Find all equilibria.
(b) Sketch the phase portrait.
(c) Determine whether each equilibrium is stable, asymptotically stable, or unstable.

a) Equilibria $\Leftrightarrow \frac{dy}{dt} = 0 \Leftrightarrow f(y) = 0 \Leftrightarrow y = 0, y = 2, y = -1.$

b) Phase Portrait



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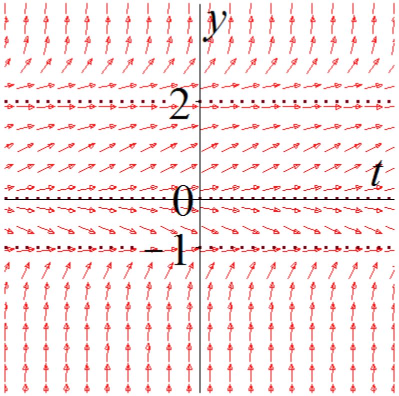
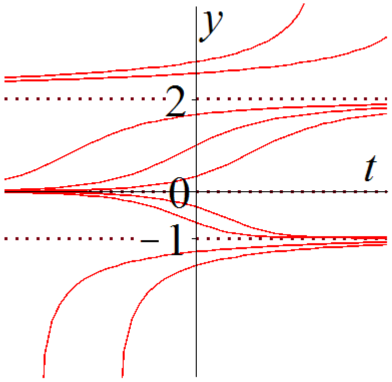
Direction Field



Solution Graphs

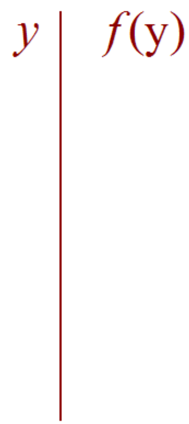


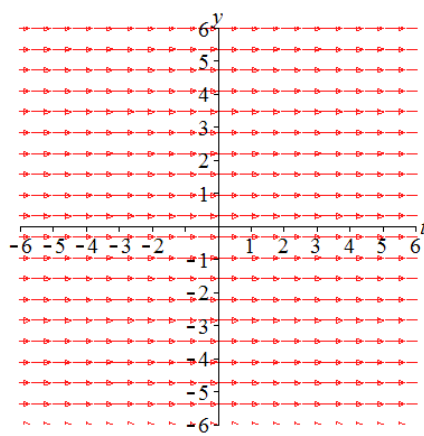
Example: $\frac{dy}{dt} = 0.$

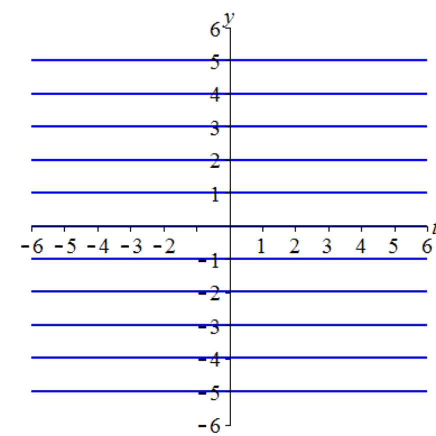
(a) Find the general solutions.
(b) Sketch the phase portrait, slope field, and solution graphs $y(t)$ vs t.
(c) Determine whether each equilibrium is stable, asymptotically stable, or unstable.

a) $\frac{dy}{dt} = 0 \Leftrightarrow y = C$. All solutions are time-independent, i.e., equilibria.

b)



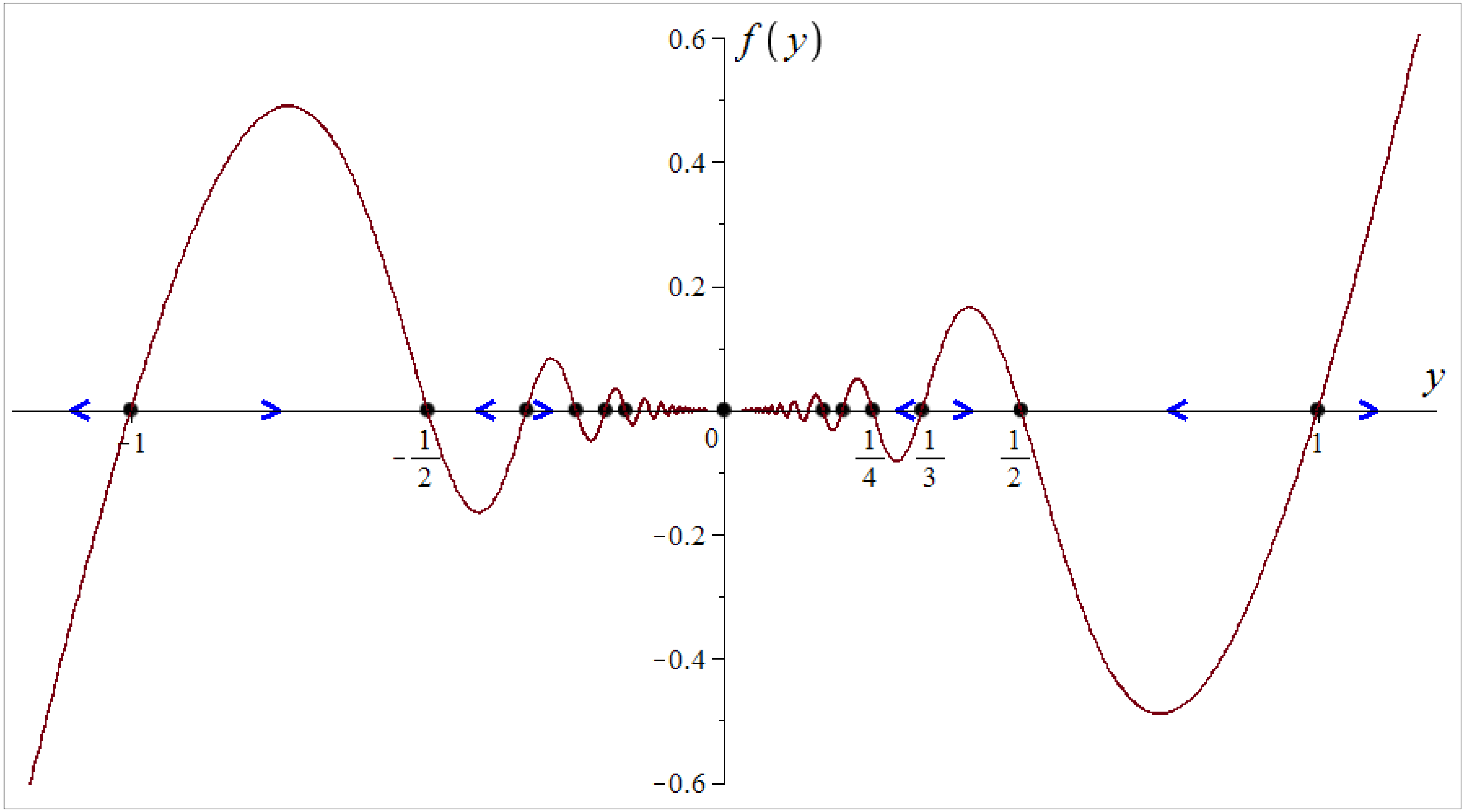




c) $y = -1$ is stable, but not asymptotically stable.
$y = 2$ is stable, but not asymptotically stable.
Every equilibrium $y = C$ is stable, but not asymptot. stable.

Example: $\frac{dy}{dt} = f(y)$ where $f(y) = \begin{cases} y^2 \sin(\pi/y) & y \neq 0 \\ 0 & y = 0 \end{cases}$

Diff eq has infinitely many equilibria near 0: $y = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{5}, \cdots\cdots$



Equilibrium $y = 0$ is stable, but not asymptotically stable.