Phase Portraits of 1-D Autonomous Equations

In each of the following problems [1]-[5]: (a) find all equilibrium solutions; (b) determine whether each of the equilibrium solutions is stable, asymptotically stable or unstable; and (c) sketch the phase portrait.

[1] $\dfrac{dP}{dt} = P(P^2 - 1)(P - 3)$.

[2] $\dfrac{dy}{dt} = -(y - 1)(y - 3)^2$.

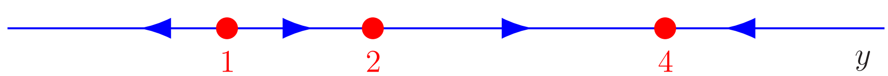
[3] $\dfrac{dy}{dt} = \sin(\pi y)$.

[4] $\dfrac{dx}{dt}(t) = \sin^2(\pi x(t))$.

[5] $\dfrac{dy}{dt} = f(y)$, where the function $f(y)$ is piecewise defined by:

$$f(y) = \begin{cases} 2y & y \le 0, \\ 0 & 0 < y < 1, \\ 1 - y & y \ge 1. \end{cases}$$

[6] An equation $\dfrac{dy}{dt} = f(y)$ has the following phase portrait.

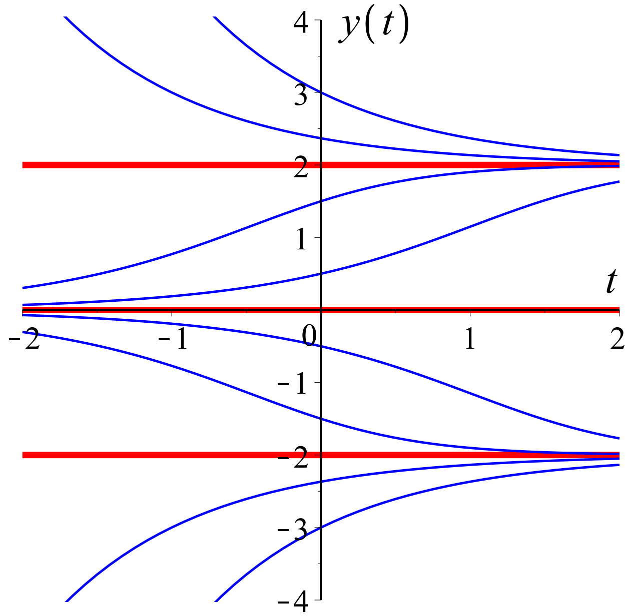


(a) Find all equilibrium solutions.

(b) Determine whether each of the equilibrium solutions is stable, asymptotically stable or unstable.

(c) Graph the solutions $y(t)$ vs t, for the initial values $y(1.4) = 0$, $y(0) = 0.5$, $y(0) = 1$, $y(0) = 1.1$, $y(0) = 1.5$, $y(-0.5) = 1.5$, $y(0) = 2$, $y(0) = 2.5$, $y(0) = 3$, $y(0) = 3.5$, $y(0) = 4$, $y(0) = 4.5$, $y(-1) = 4.5$. (Without further quantitative information about the equation and the solution formula, it's clearly impossible to draw accurate graphs of $y(t)$ vs t. Here, try to sketch graphs qualitatively to show the correct dynamic properties. The point is that a great deal of info about solution dynamics can be read off from one simple figure of phase portrait.)

[7] Several solution graphs $y(t)$ vs t are given below, for an equation $\dfrac{dy}{dt} = f(y)$.



(a) Find all equilibrium solutions in the interval $-4 < y < 4$;

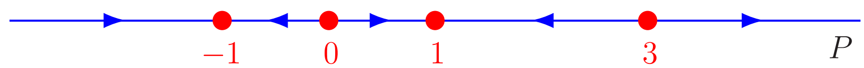
(b) Determine whether each of the above equilibrium solutions is stable, asymptotically stable or unstable;

(c) Sketch phase portrait on the interval $-4 < y < 4$.

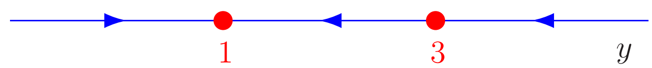
(See next page for answers)

Answers:

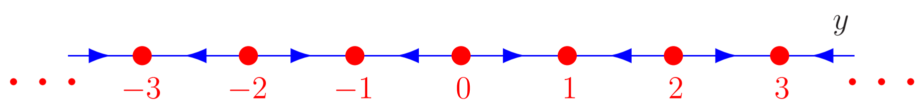
[1] There are four equilibrium solutions: $P = -1, 0, 1, 3$. The equilibria $P = -1$ and $P = 1$ are asymptotically stable. The equilibria $P = 0$ and $P = 3$ are unstable.



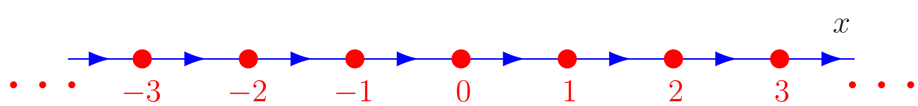
[2] There are two equilibrium solutions: $y = 1, 3$. The equilibrium $y = 1$ is asymptotically stable. The equilibrium $y = 3$ is unstable.



[3] There are infinitely many equilibrium solutions: any integer is an equilibrium. Among these equilibria, odd integers $y = \pm 1, \pm 3, \pm 5, \cdots$ are asymptotically stable, while even integers $y = 0, \pm 2, \pm 4, \pm 6, \cdots$ are unstable.



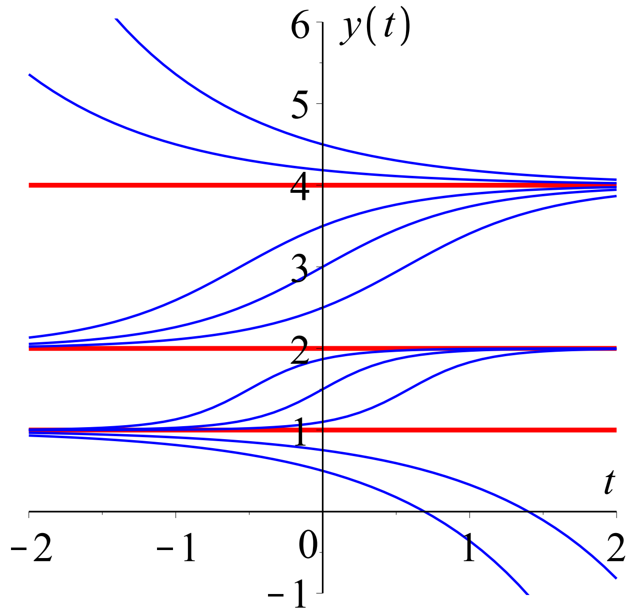
[4] There are infinitely many equilibrium solutions: any integer is an equilibrium. All equilibria are unstable.



[5] There are infinitely many (actually a continuum of) equilibrium solutions: each point y in the closed interval $0 \leq y \leq 1$ is an equilibrium. The equilibrium $y = 0$ is unstable. All other equilibria $0 < y \leq 1$ are stable but not asymptotically stable.

[6] There are three equilibria: $y = 1, 2, 4$. The equilibrium $y = 4$ is asymptotically stable. The equilibria $y = 1$ and $y = 2$ are unstable.

A rough sketch of the solution graphs is given below.



Besides the monotone properties and dynamic behavior of the solutions, also note that the solution graphs between $2 < y < 4$ should be all congruent. Indeed, they are horizontal translations of each other. This also holds for each of the following intervals: $1 < y < 2$, $4 < y < \infty$, and $-\infty < y < 1$.

[7] There are three equilibria: $y = -2, 0, 2$. The equilibria $y = -2$ and $y = 2$ are asymptotically stable. The equilibrium $y = 0$ is unstable.