Linear Approximating Differential Equations in 1-D

In each of the following problems [1]-[2]:

(a) find all equilibrium solutions of the equation $(*)$;

(b) for each equilibrium point, write down the linear approximating equation near the equilibrium and determine whether the equilibrium is stable, asymptotically stable or unstable with respect to the linear approximating equation;

(c) try to use the linear stability/instability obtained in (b) to determine whether each of the equilibria is stable, asymptotically stable or unstable with respect to the nonlinear equation $(*)$;

(d) if the linear approximation obtained in (b) was not enough to determine the stability of an equilibrium with respect to the nonlinear equation $(*)$, use other methods to determine whether the equilibrium is stable, asymptotically stable or unstable with respect to the nonlinear equation $(*)$

[1] $(*)$ $\dfrac{dy}{dt} = y(y-1)(y+2)$

[2] $(*)$ $\dfrac{dy}{dt} = -y(y-1)(y+2)^2$

In each of the following problems [3]-[4]:

(a) verify that $y = 1$ is an equilibrium;

(b) give the linear approximating equation for for $y \approx 1$;

(c) determine whether $y = 1$ is stable, asymptotically stable or unstable with respect to the nonlinear equation $(*)$.

[3] $(*)$ $\dfrac{dy}{dt} = 2y - 1 + \cos(\pi y).$

[4] $(*)$ $\dfrac{dy}{dt} = 2y - 2 + \sin(\pi y).$

(See next page for answers)

Answers:

[1] (a) There are three equilibria: $y = -2, 0, 1$.

(b)
- Near $y = -2$: the linear approximating equation is (∗∗) $\dfrac{dy}{dt} = 6(y+2)$.
  The equilibrium $y = -2$ is unstable with respect to the lin approx eq (∗∗).
- Near $y = 0$: the linear approximating equation is (∗∗) $\dfrac{dy}{dt} = -2y$.
  The equilibrium $y = 0$ is asymptotically stable w.r.t. the lin approx eq (∗∗).
- Near $y = 1$: the linear approximating equation is (∗∗) $\dfrac{dy}{dt} = 3(y-1)$.
  The equilibrium $y = 1$ is unstable with respect to the lin approx eq (∗∗).

(c) Since each of the linear approximating equations in (b) is non-degenerate, the non-linear dynamics near the equilibrium can be qualitatively determined by the linear dynamics.

The equilibria $y = -2$ and $y = 1$ are unstable with respect to the nonlin eq (∗).

The equilibrium $y = 0$ is asymptotically stable w.r.t. the nonlin eq (∗).

(d) No need to consider.

[2] (a) There are three equilibria: $y = -2, 0, 1$.

(b)
- Near $y = -2$: the linear approximating equation is (∗∗) $\dfrac{dy}{dt} = 0$.
  The equilibrium $y = -2$ is stable but not asymptotically stable, with respect to the lin approx eq (∗∗).
- Near $y = 0$: the linear approximating equation is (∗∗) $\dfrac{dy}{dt} = 4y$.
  The equilibrium $y = 0$ is unstable with respect to the lin approx eq (∗∗).
- Near $y = 1$: the linear approximating equation is (∗∗) $\dfrac{dy}{dt} = -9(y-1)$.
  The equilibrium $y = 1$ is asymptotically stable w.r.t. the lin approx eq (∗∗).

(c)
- The linear approximations are sufficient to determine the nonlinear dynamics near $y = 0$ and near $y = 1$ on the qualitatively level.
  The equilibrium $y = 0$ is unstable with respect to the nonlin eq (∗).
  The equilibrium $y = 1$ is asymptotically stable w.r.t. the nonlin eq (∗).
- On the other hand, the linear approximating equation near $y = -2$ is degenerate. The linear approximation is insufficient to determine the nonlinear dynamics near $y = -2$.

(d) For $y = -2$, the stability/instability w.r.t. the nonlinear equation (∗) can be determined by studying the sign changes of the nonlinear term $f(y) = -y(y-1)(y+2)^2$ for $y \approx -2$.

Answer: The equilibrium $y = -2$ is unstable w.r.t. the nonlinear equation (∗).

[3] (a) Let $f(y) = 2y - 1 + \cos(\pi y)$. Verify that $f(1) = 0$.

(b) $\dfrac{dy}{dt} = 2(y-1)$

(c) $y = 1$ is unstable with respect to the nonlin eq $(*)$.

[4] (a) Let $f(y) = 2y - 2 + \sin(\pi y)$. Verify that $f(1) = 0$.

(b) $\dfrac{dy}{dt} = (2-\pi)(y-1)$

(c) $y = 1$ is asymptotically stable with respect to the nonlin eq $(*)$.