Full Rank Matrix. Inverse Matrix

Rank and Nullity:

$$\begin{aligned}
\text{rank}(A) &= \dim(\text{Range of } A) \\
&= \dim(\text{Column Space of } A) \\
&= \dim(\text{Row Space of } A) \\
&= \#\text{ of pivots in the echelon form of } A \\
&= \#\text{ of nonzero rows in the echelon form of } A \\
&= \text{the maximal number of linearly independent columns in } A \\
&= \text{the maximal number of linearly independent rows in } A.
\end{aligned}$$

$$\text{nullity}(A) = \dim(\text{Nullspace of } A).$$

- If A is an $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = n$.

DEFINITION: Let A be a square matrix of size n.
An $n \times n$ matrix B is called the *inverse matrix* of A if it satisfies

$$AB = BA = I_n.$$

The inverse of A is denoted by A^{-1}.
If A has an inverse, A is said to be *invertible* or *nonsingular*.
If A has no inverses, it is said to be *not invertible* or *singular*.

HOW TO COMPUTE?

Row reduce $\begin{bmatrix} A & \vdots & I_n \end{bmatrix}$.

Case 1: The row echelon form becomes $\begin{bmatrix} I_n & \vdots & B \end{bmatrix}$.
In this case, the matrix B on the right half equals A^{-1}.

Case 2: The left half has one entire row equal to zero.
In this case, the matrix A is not invertible.

EXERCISES:

[1] Suppose that a 4×6 matrix A has rank 3.

(a) Find the nullity of A.

(b) The range of A is

(i) 0 (ii) $\mathbb{R}^4$ (iii) $\mathbb{R}^6$ (iv) none of the above.

(c) Does $A\vec{\mathbf{x}} = 0$ have no solution, infinitely many solutions, or one solution?

(d) *True or False?* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is always solvable for any vector $\vec{\mathbf{b}}$ in $\mathbb{R}^4$.

(e) *True or False?* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has at most one solution.

(f) *True or False?* The columns of A are linearly independent.

[2] Suppose that a 4×6 matrix A has rank 4.

(a) Find the nullity of A.

(b) The range of A is

(i) 0 (ii) $\mathbb{R}^4$ (iii) $\mathbb{R}^6$ (iv) none of the above.

(c) Does $A\vec{\mathbf{x}} = 0$ have no solution, infinitely many solutions, or one solution?

(d) *True or False?* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is always solvable for any vector $\vec{\mathbf{b}}$ in $\mathbb{R}^4$.

(e) *True or False?* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has at most one solution.

(f) *True or False?* The columns of A are linearly independent.

[3] Suppose that A is a 6×4 matrix such that $A\vec{\mathbf{x}} = 0$ has only one solution $\vec{\mathbf{x}} = 0$.

(a) Find the nullity and rank of A.

(b) The range of A is

(i) 0 (ii) $\mathbb{R}^4$ (iii) $\mathbb{R}^6$ (iv) none of the above.

(c) *True or False?* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is always solvable for any vector $\vec{\mathbf{b}}$ in $\mathbb{R}^6$.

(d) *True or False?* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has at most one solution.

(e) *True or False?* The columns of A are linearly independent.

[4] Suppose that a 6×6 matrix A has rank 6.

(a) Find the nullity of A.

(b) The range of A is

(i) 0 (ii) $\mathbb{R}^6$ (iii) $\mathbb{R}^{36}$ (iv) none of the above.

(c) Does $A\vec{\mathbf{x}} = 0$ have no solution, infinitely many solutions, or one solution?

(d) *True or False?* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is always solvable for any vector $\vec{\mathbf{b}}$ in $\mathbb{R}^6$.

(e) *True or False?* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has at most one solution.

(f) *True or False?* The columns of A are linearly independent.

(g) *True or False?* Matrix A is invertible.

[5] For each given matrix, determine whether the matrix is invertible. If it is invertible, find its inverse matrix.

$$\text{(a)} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 1 & 2 & -4 \\ 3 & 1 & 3 \\ 7 & -1 & 17 \end{bmatrix}, \quad \text{(d)} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix},$$

$$\text{(e)} \begin{bmatrix} -1 & 1 & -1 & 2 \\ 1 & -2 & 3 & -3 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 2 & -2 \end{bmatrix}, \quad \text{(f)} \begin{bmatrix} 0 & 1 & 2 & 4 \\ 2 & 1 & 1 & 3 \\ 1 & 3 & 4 & 10 \\ 0 & 0 & 2 & 2 \end{bmatrix}.$$

[6] Given the fact that matrix $A = \begin{bmatrix} -1 & 1 & -1 & 2 \\ 1 & -2 & 3 & -3 \\ -1 & 1 & 2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$ has inverse $A^{-1} = \begin{bmatrix} 3 & -1 & -1 & 4 \\ -3 & -1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 4 & 0 & -1 & 3 \end{bmatrix}$, solve the equation $A\vec{\mathbf{x}} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, by carrying out a matrix multiplication.

Turn over for the answers

Answers:

[1] (a) 3 (b) iv (c) Infinitely many solutions (d) False (e) False (f) False

[2] (a) 2 (b) ii (c) Infinitely many solutions (d) True (e) False (f) False

[3] (a) Nullity=0, Rank=4 (b) iv (c) False (d) True (e) True

[4] (a) 0 (b) ii (c) One solution $\vec{\mathbf{x}} = 0$ (d) True (e) True (f) True (g) True

[5] (a) $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

(b) A is singular

(c) A is singular

(d) $A^{-1} = \begin{bmatrix} 1 & -3/2 & 1/2 \\ 1 & -1 & 0 \\ -2 & 7/2 & -1/2 \end{bmatrix}$

(e) $A^{-1} = \begin{bmatrix} -1 & 1 & 1 & -2 \\ -2 & 0 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 \end{bmatrix}$

(f) A is singular

[6] $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ -5 \\ 1 \\ 4 \end{bmatrix}$