The Riemann Zeta Function
Part 2: pole and zeros

THEOREM (Pole and Trivial Zeros of $\zeta(s)$):

(a) $\zeta(s)$ *is holomorphic in* $\mathbb{C} \setminus \{1\}$ *and has a simple pole at* $s = 1$ *with residue* $= 1$*; in other words,* $\zeta(s) = \dfrac{1}{s-1} + \text{(an entire function)}$.

(b) *The only zeros of* $\zeta(s)$ *with* $\operatorname{Re} s \leq 0$ *or* $\operatorname{Re} \geq 1$ *are the simple zeros at negative even integers* $s = -2, -4, \cdots$.

The mail tools of the proof are the Euler product and Riemann's functional equation.

Euler Product: $\displaystyle \zeta(s) = \prod_{p:\text{prime}} \frac{1}{1-p^{-s}} \qquad (\operatorname{Re} s > 1).$

Riemann's Functional Equation: $\zeta(s) \equiv 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s).$

Outline of a Proof of the Theorem

(i) The only pole of $\zeta(s)$ is at $s = 1$, where the residue $= 1$.
Keys: Evaluate $G(1)$ directly by a residue calculation, plus some basic properties of the Gamma function.

(ii) $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$.
Keys: The Euler product.

(iii) The only zeros of $\zeta(s)$ in $\operatorname{Re} s < 0$ are the simple zeros at negative even integers.
Keys: (ii) and Riemann's functional equation in addition to the basics of the Gamma function. (Lemma 1 gives an alternative method of proving the vanishing of $\zeta(-2n)$ by evaluating $G(-2n)$ directly.)

(iv) $\zeta(s)$ has no zeros on the vertical line $\operatorname{Re} s = 1$.
Keys: The log derivative of the Euler product and an elementary trick (Hadamard-de la Vallée Poussin).

(v) $\zeta(s)$ has no zeros on the vertical line $\operatorname{Re} s = 0$.
Keys: This follows from (i), (iv), and Riemann's functional equation immediately.

The proof of the Euler product is relatively easy. Simply expand $(1-p^{-s})^{-1}$ into $\sum_{k=0}^{\infty} p^{-ks}$ and multiply these series together.

Next in this note we will prove (i)(ii)(iii). A proof of Riemann's functional equation based on the residue analysis will be given at the end of this note.

Step (iv) will be done later in Part 3.

LEMMA 1: *(a) $G(n) = 0$ for integer $n \geq 2$.*

(b) $G(1) = 2\pi i$.

(c) $G(-n) = 2\pi i \dfrac{a_{n+1}}{(n+1)!}$ for integers $n \geq 0$. Here a_k are the Taylor coefficients in

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k.$$

In particular, $G(-2) = G(-4) = \cdots = 0$.

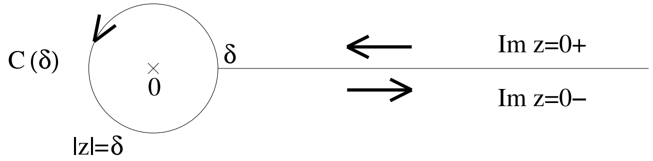
(d) The only pole of $\zeta(s)$ is a simple pole at $s = 1$ with residue $= 1$.

(e) $\zeta(-2) = \zeta(-4) = \cdots = 0$.

Proof: We have shown in Part 1 that:

$$G(s) = \int_{C(\delta)} \frac{z^{s-1}}{e^z - 1} dz = \int_{|z|=\delta} \frac{z^{s-1}}{e^z - 1} dz + (e^{i2\pi s} - 1) \int_{\delta}^{\infty} \frac{x^{s-1}}{e^x - 1} dx,$$

where the integration contour $C(\delta)$ with $0 < \delta < 2\pi$ is:



(Recall also that $G(s)$ is independent of δ.)

Proof of (a)(b)(c): Let s in the above formula be an integer: $s = n$. The second integral on the right hand side vanishes, and thus

$$G(n) = \int_{|z|=\delta} \frac{z^{n-1}}{e^z - 1} dz = 2\pi i \operatorname{Res} \left(\frac{z^{n-1}}{e^z - 1}; 0 \right).$$

The rest is easy. Let's just remark that the function

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2} \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}$$

is an even function. This implies

$$a_1 = -\frac{1}{2}, a_3 = a_5 = a_7 = \cdots = 0.$$

The first few a_{2k} are

$$a_0 = 1, a_2 = \frac{1}{6}, a_4 = -\frac{1}{30}, a_6 = \frac{1}{42}, a_8 = -\frac{1}{30}, a_{10} = \frac{5}{66}, \cdots,$$

and they are connected to the so-called Bernoulli numbers: $a_{2k} = (-1)^{k-1} B_{2k}$.

Proof of (d)(e): Apply the results in (a)(b)(c) to

$$\zeta(s) = \frac{G(s)}{(e^{i2\pi s} - 1)\Gamma(s)} \qquad (s \in \mathbb{C}).$$

From this definition of $\zeta(s)$, we see that the only possible singularities of $\zeta(s)$ are the zeros of $e^{i2\pi s}-1$, that is, $s =$ integers. Recall that for positive integers $n \geq 1$, $\Gamma(n) = (n-1)!$, and for nonpositive integers $s = -n \leq 0$, $\Gamma(s)$ has a simple pole at $s = -n$ with residue $\operatorname{Res}(\Gamma; -n) = (-1)^n/n!$. The statements (d)(e) will follow readily. ▮

LEMMA 2: $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$.

Proof: We use the Euler product formula.
Write $s = \sigma + it$ with $\sigma > 1$ and $t \in \mathbb{R}$. Recall that for all real x,

$$e^x - (1+x) = \int_0^x (x-u)e^u du \geq 0.$$

(This is the integral form of the remainder for the Taylor approximation $e^x \approx 1+x$.) We have

$$\left|1-p^{-s}\right| \leq 1 + \left|p^{-s}\right| = 1 + p^{-\sigma} \leq \exp\left(p^{-\sigma}\right).$$

Use this in the Euler product:

$$\begin{aligned} |\zeta(s)| &= \prod_{p:\text{prime}} \left|1-p^{-s}\right|^{-1} \\ &\geq \prod_{p:\text{prime}} \exp\left(-p^{-\sigma}\right) = \exp\left(-\sum_{p:\text{prime}} p^{-\sigma}\right) \\ &\geq \exp\left(-\sum_{n=2}^{\infty} n^{-\sigma}\right) > 0. \qquad ▮ \end{aligned}$$

LEMMA 3: The only zeros of $\zeta(s)$ in $\operatorname{Re} s < 0$ are the simple zeros at negative even integers $s = -2, -4, \cdots$.

Proof: Let $\operatorname{Re} s < 0$. We are going to use Riemann's functional equation:

$$\zeta(s) \equiv 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s).$$

First, among the factors on the right hand side, $\zeta(1-s) \neq 0$ by Lemma 2. Moreover, $\zeta(1-s)$ is holomorphic in $\operatorname{Re} s < 0$.
Second, $\Gamma(z)$ has no zeros in $\mathbb{C}$, since by definition

$$\Gamma(z) = \frac{1}{\text{an entire function}}.$$

We also know that $\Gamma(z)$ has simple poles only at nonpositive integers. Thus, $\Gamma(1-s) \neq 0$ and is holomorphic in $\operatorname{Re} s < 0$.

Thus, the zeros of $\zeta(s)$ with $\operatorname{Re} s < 0$ are exactly the zeros of $\sin(\pi s/2)$ with $\operatorname{Re} s < 0$, that is, $s = -2, -4, \cdots$.

Moreover, differentiating the functional equation at $s = -2n$, we obtain

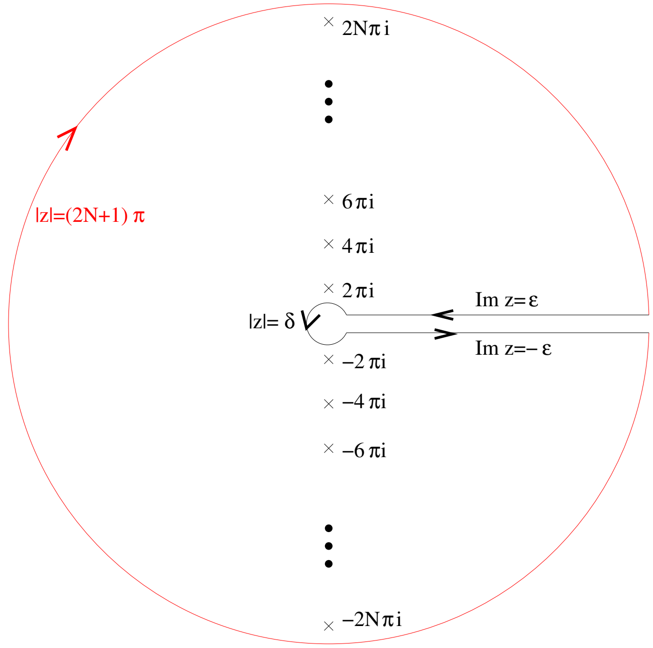
$$\zeta'(-2n) = (-1)^n 2^{-2n-1} \pi^{-2n} \Gamma(2n+1) \zeta(2n+1) \neq 0.$$

Therefore, negative even integers $s = -2n$ are simple zeros of $\zeta(s)$. ▮

Now we give

Proof of Riemann's Functional Equation: It suffices to prove the functional equation for $\mathrm{Re}\, s < 0$.

Take $0 < \varepsilon < \delta < 2\pi$ and a (large) positive integer N. Integrate $g(s,z) = z^{s-1}/(e^z - 1)$ on the following closed contour $C(\delta, \varepsilon, N)$ in the z-plane:



which consists of the black and red paths in the above figure.

Notice that the poles of $g(s,z)$ (as a function of z) enclosed by the closed contour $C(\delta, \varepsilon, N)$ are $2n\pi i$ with $n = \pm 1, \pm 2, \cdots, \pm N$. By the residue theorem,

$$\int_{C(\delta,\varepsilon,N)} g(s,z)dz = -2\pi i \sum_{-N \le n \le N, n \ne 0} \mathrm{Res}\left(\frac{e^{(s-1)\log z}}{e^z - 1}; z = 2n\pi i\right)$$

(the negative sign in front of $2\pi i$ is due to the orientation of the contour)

$$= -2\pi i \left[\sum_{n=1}^{N} e^{(s-1)[\ln(2n\pi)+i\pi/2]} + \sum_{n=1}^{N} e^{(s-1)[\ln(2n\pi)+i3\pi/2]}\right]$$

$$= -2\pi i \sum_{n=1}^{N} \left[(2n\pi)^{s-1} e^{i(s-1)\pi/2} + (2n\pi)^{s-1} e^{i3(s-1)\pi/2}\right]$$

$$= -i(2\pi)^s \left[e^{i(s-1)\pi/2} + e^{i3(s-1)\pi/2}\right] \sum_{n=1}^{N} n^{s-1}$$

$$\longrightarrow 2i(2\pi)^s e^{i\pi s} \sin(\pi s/2)\zeta(1-s) \qquad \text{as } N \to \infty.$$

On the other hand, taking the limit as $N \to \infty$, we have

$$\int_{\text{black}} g(s,z)dz \to \int_{C(\delta,\varepsilon)} g(s,z)dz = G(s) = (e^{2\pi i s}-1)\Gamma(s)\zeta(s),$$
$$\int_{\text{red}} g(s,z)dz \to 0.$$

The proof of the latter is based on the following uniform estimate on the red path:

$$\max_{|z|=(2N+1)\pi} \frac{1}{|e^z-1|} \le M,$$

where $M > 0$ is a positive constant independent of N. Using this we have

$$\left|\int_{\text{red}} g(s,z)dz\right| \le [(2N+1)\pi]^{\operatorname{Re} s}\, M \int_0^{2\pi} e^{-\operatorname{Im} s\theta} d\theta.$$

When $\operatorname{Re} s < 0$, the right hand side decays to 0 as $N \to \infty$. Passing to the limit, we obtain

$$(e^{2\pi i s}-1)\Gamma(s)\zeta(s) = 2i(2\pi)^s e^{i\pi s}\sin(\pi s/2)\zeta(1-s),$$

and hence

$$\zeta(s) = \frac{2i(2\pi)^s e^{i\pi s}\sin(\pi s/2)\zeta(1-s)}{(e^{2\pi i s}-1)\Gamma(s)} = (2\pi)^s\zeta(1-s)\frac{\sin(\pi s/2)}{\sin(\pi s)\Gamma(s)}.$$

Recalling an indentity for the Gamma function:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

we finally obtain

$$\zeta(s) = (2\pi)^s\zeta(1-s)\frac{\sin(\pi s/2)\Gamma(1-s)}{\pi} \qquad (\operatorname{Re} s < 0).$$

The proof is complete. ▮

EXERCISES

1. Show that $\dfrac{1}{\zeta(s)} = \displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ for $\operatorname{Re} s > 1$. where $\mu(n)$ is defined as follows:

$$\mu(n) = \begin{cases} 1 & n = 1, \\ (-1)^k & n = p_1 \cdots p_k \text{ with distinct primes } p_1, \cdots, p_k, \\ 0 & \text{otherwise.} \end{cases}$$

2. Evaluate $\zeta(0)$ and $\zeta'(0)$.

3. For positive integer N, let M_N be

$$M_N = \max_{|z|=(2N+1)\pi} \frac{1}{|e^z-1|}.$$

   Show that $\limsup_{N\to\infty} M_N < \infty$. (This estimate was used in our proof of Riemann's functional equation.)