Infinity as an Isolated Singularity

We have so far discussed isolated singularities of holomorphic functions in the complex plane. In this note, we extend the study to the case where $z = \infty$ is an isolated singularity.

Definition (Isolated Singularity at Infinity): The point at infinity $z = \infty$ is called an *isolated singularity* of $f(z)$ if $f(z)$ is holomorphic in the exterior of a disk $\{z \in \mathbb{C} : |z| > R\}$.

This is quite natural, since through the stereographic projection the region $\{z \in \mathbb{C} : |z| > R\}$ corresponds to a punctured disk on the sphere centered at the north pole.

Notice also that $z = \infty$ is an isolated singularity of $f(z)$ if and only if $z = 0$ is an isolated insgularity of $f(1/z)$. Furthermore, we use the following definitions to classify the singularities at $z = \infty$.

Definition (Classifications): Let $z = \infty$ be an isolated singularity of $f(z)$.

(a) $f(z)$ has a *removable singularity* at $z = \infty$ if $f(1/z)$ has a removable singularity at $z = 0$.

(b) $f(z)$ has a *pole of order* $m \geq 1$ at $z = \infty$ if $f(1/z)$ has a pole of order $m \geq 1$ at $z = 0$.

(c) $f(z)$ has an *essential singularity* at $z = \infty$ if $f(1/z)$ has an essential singularity at $z = 0$.

Proposition (Laurent Series): *We easily obtain the following results:*

(a) *If $z = \infty$ is an isolated singularity of $f(z)$, then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad (|z| > R),$$

where R is a positive number.

(b) *If $z = \infty$ is a removable singularity of $f(z)$, then $a_n = 0$ for all $n > 0$:*

$$f(z) = \sum_{n=-\infty}^{0} a_n z^n \quad (|z| > R).$$

(c) *If $z = \infty$ is a pole of order $m \geq 1$ of $f(z)$, then $a_m \neq 0$ and $a_n = 0$ for all $n > m$:*

$$f(z) = \sum_{n=-\infty}^{m} a_n z^n \quad (|z| > R).$$

(d) *If $z = \infty$ is an essential singularity of $f(z)$, then $a_n \neq 0$ for infinitely many positive integers n.*

Definition (Zero at Infinity): It is also natural to call $z = \infty$ a zero of multiplicity $m \geq 1$ of $f(z)$ if $f(1/z)$ can be extended to a holomorphic function $g(z)$ on a disk $B(0, \delta)$ and $z = 0$ is a zero of multiplicity m of $g(z)$.

An equivalent condition is: In the above Laurent series expansion near $z = \infty$, $a_{-m} \neq 0$ and $a_n = 0$ for all $n > -m$:

$$f(z) = \sum_{n=-\infty}^{-m} a_n z^n \quad (|z| > R).$$

Theorem (Entire Functions Behaving Good at Infinity are Polynomials): *Let $f(z)$ be an entire function (that is, $f(z)$ is holomorphic in the entire complex plane $\mathbb{C}$).*

(a) *If $z = \infty$ is a removable singularity of $f(z)$, then $f(z)$ is a constant.*

(b) *If $z = \infty$ is a pole of order $m \geq 1$ of $f(z)$, then $f(z)$ is a polynomial of degree m.*

Definition (Transcendental Entire Functions): An entire function $f(z)$ is called a *transcendental entire function* if $z = \infty$ is an essential singularity of $f(z)$. In view of the above theorem, a transcendental entire function is an entire function that is not a polynomial.

Examples: (i) $\cos z$, (ii) $\sin(\pi z)$, (iii) e^{-z^2} are transcendental entire functions. There are other important and more sophicated examples: (iv) the Bessel function of the first kind of order $k \geq 0$:

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} \left(\frac{z}{2}\right)^{2n+k},$$

(v) $(z-1)\zeta(z)$, where $\zeta(z)$ is the Riemann zeta function which we will discuss later in the course.

Definition (Meromorphic Functions): Let $G \subset \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be open in $\mathbb{C}_\infty$. A function f is said to be *meromorphic* in G if it is defined and holomorphic in G except for isolated singularities and all isolated singularities are either removable or poles.

The previous theorem can now be rephrased as: If f is meromorphic in $\mathbb{C}_\infty$ and has no poles in $\mathbb{C}$, then it must be a polynomial.

Rational Functions: Let $f(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z) \not\equiv 0$ are polynomials of degree m and n respectively. It is easy to see that $f(z)$ is meromorphic in $\mathbb{C}_\infty$. Moreover,

$$\begin{cases} z = \infty \text{ is a zero of order } n-m \text{ of } f(z) \text{ if } n > m; \\ z = \infty \text{ is a removable singularity of } f(z) \text{ if } n = m; \\ z = \infty \text{ is a pole of order } m-n \text{ of } f(z) \text{ if } n < m. \end{cases}$$

Conversely, we have the following:

Theorem (Meromorphic Functions on $\mathbb{C}_\infty$):
If $f(z)$ is a meromorphic function on $\mathbb{C}_\infty$, then $f(z)$ is a rational function of z.

Definition (Transcendental Meromorphic Functions): A function $f(z)$ is called a *transcendental meromorphic function* if it is meromorphic in $\mathbb{C}$ and is not a rational function.

The above theorem shows that if $f(z)$ is a transcendental meromorphic function, then

either (a) $f(z)$ has at most a finite number of poles in $\mathbb{C}$ and has an essential singularity at $z = \infty$;

or (b) there are an infinite number of poles z_n of $f(z)$ accumulating at infinity: $z_n \to \infty$.

Examples: $e^z/(1+z^2), 1/\sin z$, and $\Gamma(z)$ are transcendental meromorphic functions.
$e^z/(1+z^2)$ satisfies (a).
$1/\sin z$ satisfies (b).

The gamma function $\Gamma(z)$, which will be studied later in the course, exhibits behavior (b). It has simple poles at negative integers $z = -1, -2, ...$

Exercise:

1. Prove all claims in this note.

2. Let $R(z)$ be a rational function of z and assume $R(z) \not\equiv 0$. Show that the number of zeros of $R(z)$ in $\mathbb{C}_\infty$ equals the number of poles of $R(z)$ in $\mathbb{C}_\infty$. Here, zeros and poles are counted repeatedly according to their multiplicities and orders.

3. Suppose that $f(z)$ is holomorphic in $|z| > R$. We have seen that f can be expanded into the Laurent series:
$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad (|z| > R).$$
Define the residue of f at $z = \infty$ by:
$$\mathrm{Res}(f; \infty) = -a_{-1}.$$
Notice that even when $z = \infty$ is a removable singularity of $f(z)$, it is possible that $\mathrm{Res}(f; \infty) \neq 0$.
   (a) Show that $\mathrm{Res}(f; \infty)$ equals the residue of $-z^{-2} f(1/z)$ at $z = 0$.
   (b) Show that
   $$\mathrm{Res}(f; \infty) = -\frac{1}{2\pi i} \int_{|z|=r} f(z)dz \quad (R < r < \infty),$$
   where the circle $|z| = r$ is oriented counterclockwise.
   (c) Let f be holomorphic in $\mathbb{C}$ except for a finite number of isolated singularities $z_1, ..., z_n \in \mathbb{C}$. Show that
   $$\mathrm{Res}(f; z_1) + ... + \mathrm{Res}(f; z_n) + \mathrm{Res}(f; \infty) = 0.$$