Exercises for Taylor Series and Laurent Series

[1] Find the Taylor series of $f(z)$ expanded about the given point. Give the region where the series converges.

(a) $f(z) = 1/(z+2)$ expanded about $z = 0$.

(b) $f(z) = 1/(z+2)$ expanded about $z = 3i$.

(c) $f(z) = z^5/(z^3 - 4)$ expanded about $z = 0$.

(d) $f(z) = z \sin z$ expanded about $z = \pi/2$.

(e) $f(z) = \operatorname{Log} z$ expanded about $z = 3$.

[2] Consider the Taylor series of $f(z) = 1/(2 - \sin z)$ about the origin $z = 0$. What is the radius of convergence of this Taylor series?

[3] Give the first three nonzero terms of the Taylor series of the given function about the specified point. Find the radius of convergence of the Taylor series.

(a) $f(z) = e^{3z - z^2}$ about $z = 0$.

(b) $f(z) = \cot z = \dfrac{\cos z}{\sin z}$ about $z = \pi/2$.

(c) $f(z) = \operatorname{Log}(i + e^{2z})$ about $z = 0$.

[4] Find the Laurent series of $f(z)$ expanded about the given point:

(a) $f(z) = \dfrac{1}{(1-z)(z+2)}$ expanded about $z = 0$ for $|z| < 1$

(b) $f(z) = \dfrac{1}{(1-z)(z+2)}$ expanded about $z = 0$ for $1 < |z| < 2$

(c) $f(z) = \dfrac{1}{(1-z)(z+2)}$ expanded about $z = 0$ for $|z| > 2$

(d) $f(z) = z^3 \cos(1/z)$ expanded about $z = 0$ for $|z| > 0$.

(e) $f(z) = \sin\left(\dfrac{z}{1-z}\right)$ expanded about $z = 1$ for $|z - 1| > 0$.

(f) $f(z) = \operatorname{Log}\left(1 - \dfrac{2}{z^2}\right)$ expanded about $z = 0$ for $|z| > \sqrt{2}$.

See next page for the answers

ANSWERS

[1] (a) $f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n$ for $|z|<2$

(b) $f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2+3i)^{n+1}}(z-3i)^n$ for $|z-3i|<|-2-3i|=\sqrt{13}$

(c) $f(z)=\sum_{n=0}^{\infty}\frac{-1}{4^{n+1}}z^{3n+5}$ for $|z|<4^{1/3}$

(d) $f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\pi/2}{(2n)!}(z-\pi/2)^{2n}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(z-\pi/2)^{2n+1}$ for $|z|<\infty$

(e) $f(z)=\ln 3+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot 3^n}(z-3)^n$ for $|z-3|<3$

[2] $\sqrt{\frac{\pi^2}{4}+\left[\ln\left(2-\sqrt{3}\right)\right]^2}$

Hint: It's quite hopeless to solve the problem by finding the values of Taylor coefficients and then carefully studying their asymptotics as $n\to\infty$. Rather we should examine: What is the largest disk about the origin in which $f(z)$ is complex analytic?

[3] (a) $1+3z+\frac{7}{2}z^3$, $R=\infty$. (b) $-(z-\frac{\pi}{2})-\frac{1}{3}(z-\frac{\pi}{2})^3-\frac{2}{15}(z-\frac{\pi}{2})^5$, $R=\pi/2$.
(c) $\left(\frac{1}{2}\ln 2+i\frac{\pi}{4}\right)+(1-i)z+z^2$, $R=\pi/4$.

[4] (a) $f(z)=\sum_{n=0}^{\infty}\left(\frac{1}{3}+\frac{(-1)^n}{6\cdot 2^n}\right)z^n$ for $|z|<1$

(b) $f(z)=\sum_{n=0}^{\infty}\frac{-1}{3}z^{-1-n}+\sum_{n=0}^{\infty}\frac{(-1)^n}{6\cdot 2^n}z^n$ for $1<|z|<2$

(c) $f(z)=\sum_{n=0}^{\infty}\left(\frac{-1}{3}+\frac{(-2)^n}{3}\right)z^{-1-n}$ for $2<|z|<\infty$

(d) $f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{3-2n}$ for $0<|z|<\infty$

(e) $f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\sin 1}{(2n)!}(z-1)^{-2n}+\sum_{n=0}^{\infty}\frac{(-1)^n\cos 1}{(2n+1)!}(z-1)^{-2n-1}$ for $0<|z-1|<\infty$

(f) $f(z)=-\sum_{n=1}^{\infty}\frac{2^n}{n}z^{-2n}$ for $|z|>\sqrt{2}$