Complex Analytic Functions

[1] Determine whether or not the following functions are complex analytic:

(a) $f(z) = (z+1)/(z-1) \quad (z \neq 1)$

(b) $f(z) = |z|^2$

(c) $f(z) = e^{2y}\cos(2x) + ie^{2y}\sin(2x)$ where $x = \operatorname{Re} z, y = \operatorname{Im} z$

(d) $f(z) = e^{-y}(x\cos x - y\sin x) + ie^{-y}(y\cos x + x\sin x)$ where $x = \operatorname{Re} z, y = \operatorname{Im} z$

[2] Can each of the following functions be the real part of a complex analytic function? If yes, give such a complex analytic function. If no, explain the reason.

(a) $u(x,y) = x^2$ (b) $u(x,y) = xy$ (c) $u(x,y) = e^{2y}\cos(2x)$

[3] Prove the following:

(a) If $f(z) = u + iv$ and $\overline{f(z)} = u - iv$ are both complex analytic, then $f(z)$ must be a constant.

(b) If $f(z)$ is complex analytic and $|f(z)|$ is constant, then $f(z)$ must be a constant.

Answers:

[1] (a) Y (b) N (c) N (d) Y

[2] (a) No, since u is not harmonic.

(b) Yes. $f(z) = -\frac{i}{2}z^2$ (or $-\frac{i}{2}z^2 + iC$ with any real constant C)

(c) Yes. $f(z) = e^{-2iz}$ (or $e^{-2iz} + iC$ with any real constant C)

[3] (a) Use the Cauchy-Riemann equations.

(b) Distinguish the cases where $|f(z)| \equiv 0$ and $|f(z)| \equiv c \neq 0$. The former case is trivial. In the latter case, we have $\overline{f(z)} = c^2/f(z)$ and hence $\overline{f(z)}$ is complex analytic. The result follows from (a).