Differentiate Trig Functions

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Formulas

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x = -\frac{1}{\sin^2 x}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x = \frac{\sin x}{\cos^2 x}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x = -\frac{\cos x}{\sin^2 x}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \right) \\
&= \cos x.
\end{aligned}$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \left(\frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h} \right) \\
&= -\sin x.
\end{aligned}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x = \frac{1}{\cos^2 x}$$

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x = \frac{\sin x}{\cos^2 x}$$

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\
&= -\frac{-\sin x}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \sec x \tan x.
\end{aligned}$$

Differentiate $x^3 \cos x$.

Use the Product Rule:

$$\begin{aligned}
\frac{d}{dx}(x^3 \cos x) &= (x^3)' \cos x + x^3 (\cos x)' \\
&= 3x^2 \cos x + x^3(-\sin x) \\
&= 3x^2 \cos x - x^3 \sin x.
\end{aligned}$$

Differentiate $\tan(4x)$.

Use the Chain Rule:

$$y = \tan u, \quad u = 4x.$$

$$\begin{aligned}
\frac{d}{dx}\tan(4x) &= \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \\
&= (\sec^2 u)(4) \\
&= 4\sec^2(4x).
\end{aligned}$$

Differentiate $\sin^4 x \cos^7(3x)$.

First use the Product Rule:

$$\left[\sin^4 x \cos^7(3x)\right]' = (\sin^4 x)' \cos^7(3x) + \sin^4 x \left[\cos^7(3x)\right]'$$

To get $(\sin^4 x)'$, use the Chain Rule:

$$y = u^4, \quad u = \sin x,$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3 \cdot \cos x = 4\sin^3 x \cos x.$$

To get $\left[\cos^7(3x)\right]'$, use the Chain Rule:

$$z = v^7, \quad v = \cos w, \quad w = 3x,$$

$$\frac{dz}{dx} = \frac{dz}{dv}\frac{dv}{dw}\frac{dw}{dx} = 7v^6 \cdot (-\sin w) \cdot 3 = -21\cos^6(3x)\sin(3x).$$

$$\left[\sin^4 x \cos^7(3x)\right]' = (\sin^4 x)' \cos^7(3x) + \sin^4 x \left[\cos^7(3x)\right]'$$
$$= 4\sin^3 x \cos x \cos^7(3x) - 21\sin^4 x \cos^6(3x)\sin(3x).$$

Differentiate $\dfrac{\sin^4 x}{1+\tan 3x}$.

First use the Quotient Rule (and the Chain Rule):

$$\left[\frac{\sin^4 x}{1+\tan 3x}\right]'$$

$$= \frac{(4\sin^3 x\cos x)(1+\tan 3x) - (\sin^4 x)(3\sec^2 3x)}{(1+\tan 3x)^2}$$

$$= \frac{4\sin^3 x\cos x + 4\sin^3 x\cos x\tan 3x - 3\sin^4 x\sec^2 3x}{(1+\tan 3x)^2}$$

Differentiate $\sec(-3x + 2\cot x)$.

Use the Chain Rule:

$$y = \sec u, \quad u = -3x + 2\cot x.$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (\sec u \tan u)(-3 - 2\csc^2 x) \\ &= (-3 - 2\csc^2 x)\sec(-3x + 2\cot x)\tan(-3x + 2\cot x).\end{aligned}$$

Find the fourth derivative of $\sin x$.

$$\begin{aligned}
(\sin x)' &= \cos x \\
(\sin x)'' &= (\cos x)' = -\sin x \\
(\sin x)''' &= (-\sin x)' = -\cos x \\
(\sin x)^{(4)} &= (-\cos x)' = \sin x
\end{aligned}$$