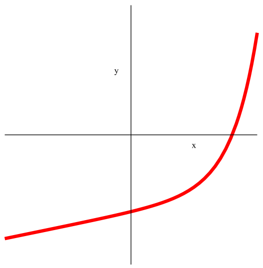
Increasing and Decreasing Functions

Xu-Yan Chen

Theorem.

If $f'(x) > 0$ on an interval (a, b),
then $f(x)$ increases on (a, b);
that is, $f(x_1) < f(x_2)$ for all $a < x_1 < x_2 < b$.



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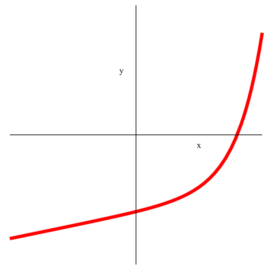

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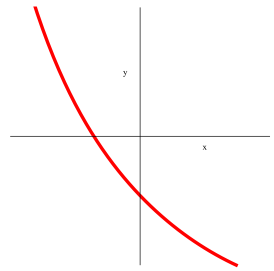

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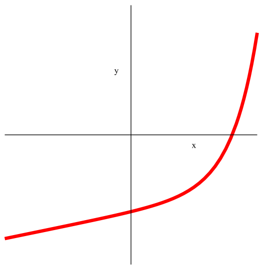

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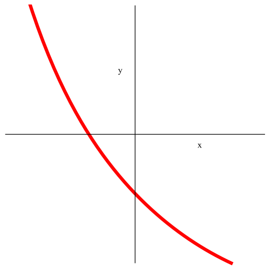

If $f'(x) = 0$ on an interval (a, b),
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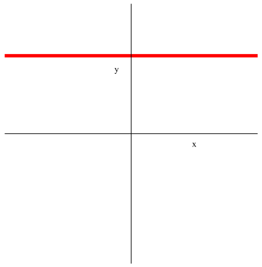

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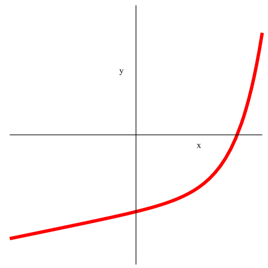

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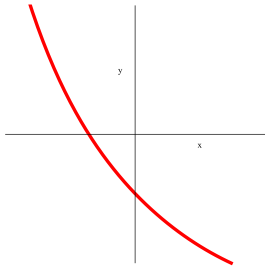

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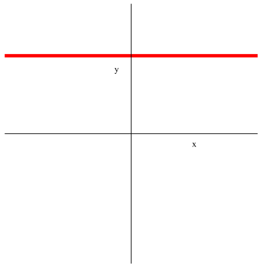

Proof. Apply the mean value theorem.

Example 1.

$f(x) = \frac{1}{3}x^3 + x - 2$

$f'(x) = x^2 + 1 > 0$ for all real x.

$f(x)$ increases on $(-\infty, \infty)$.

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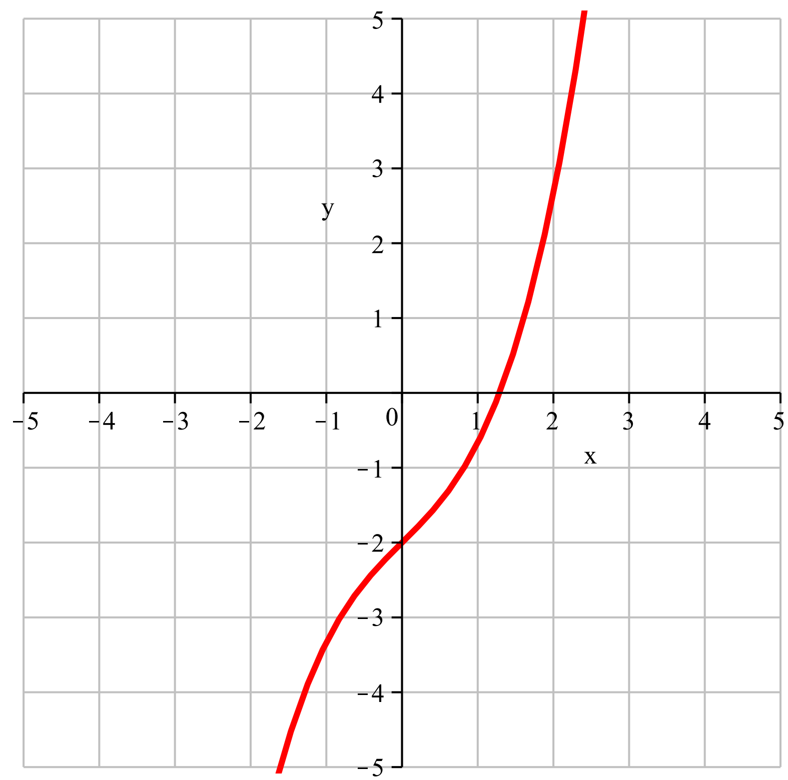

Example 2. $f(x) = -\frac{1}{12}x^3 + \frac{1}{8}x^2 + \frac{1}{2}x - 2$

$$f'(x) = -\frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{2} = -\frac{1}{4}(x+1)(x-2).$$

x		-1		2	
Sign of $f'(x)$	$-$	0	$+$	0	$-$
Behavior of $f(x)$	$\searrow$		$\nearrow$		$\searrow$

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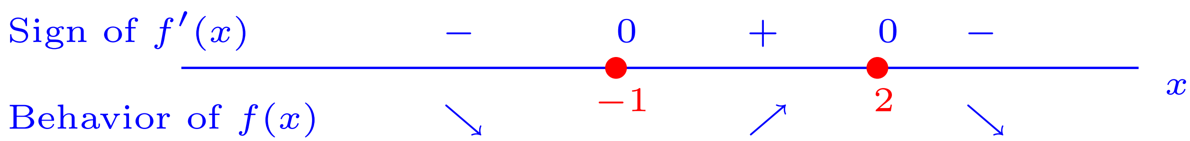

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decreases on $(-\infty, -1]$,
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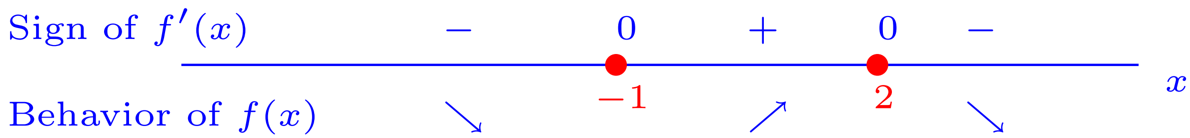

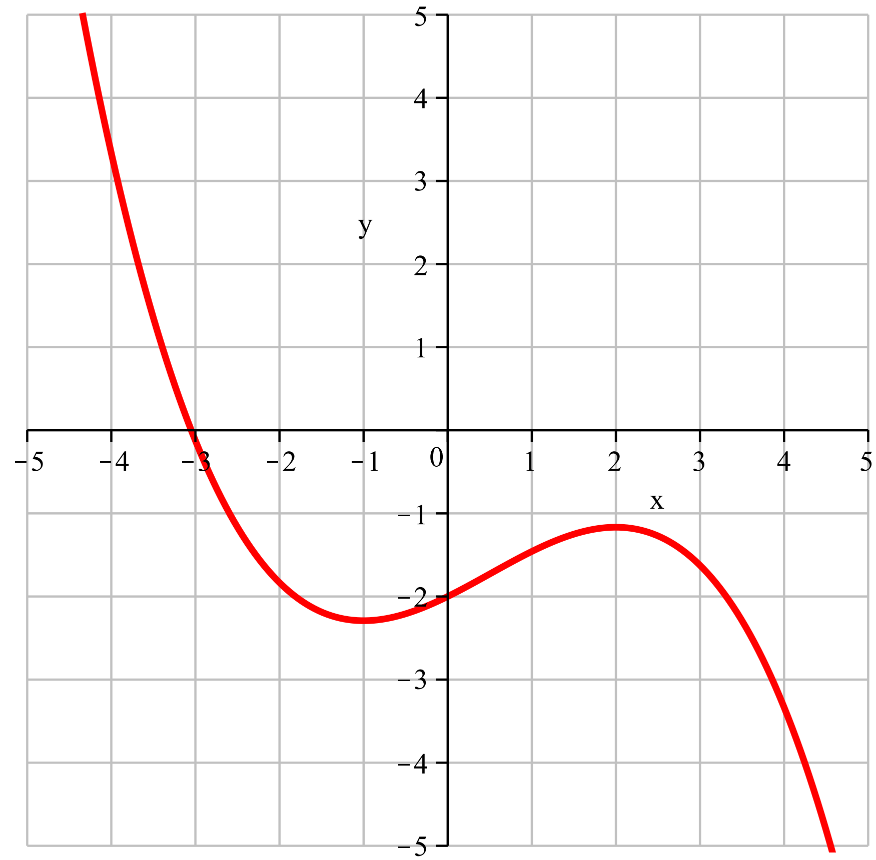

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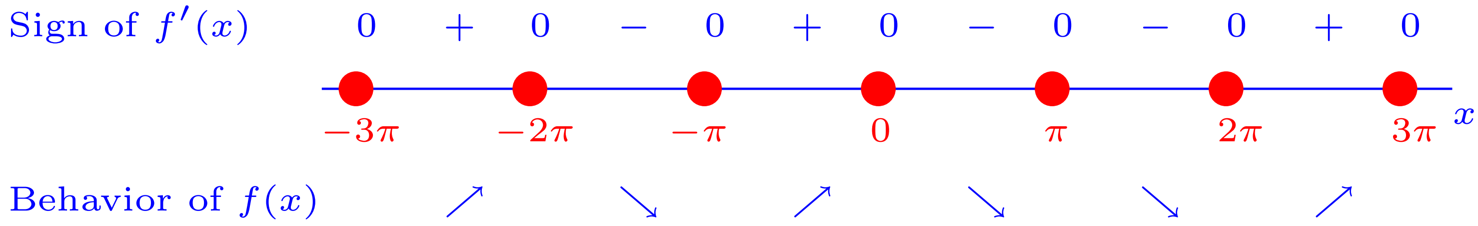
Example 3. $f(x) = -\pi + \sin x + \pi \cos x - x \cos x$ **for** $-3\pi \le x \le 3\pi$

$$f'(x) = -\pi \sin x + x \sin x = (x - \pi) \sin x.$$



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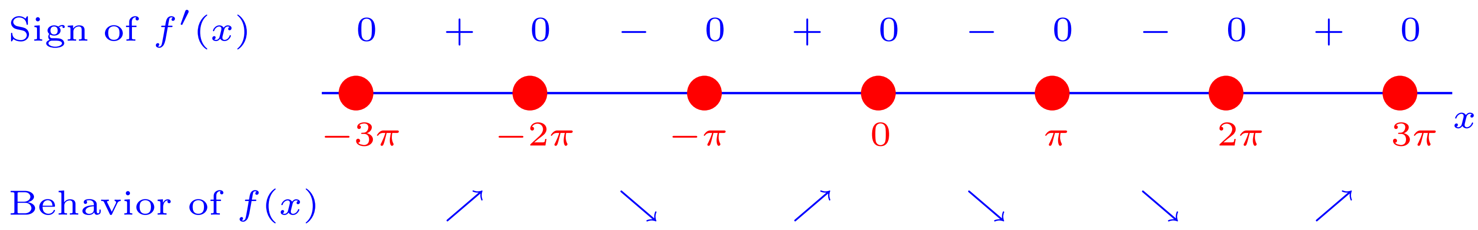

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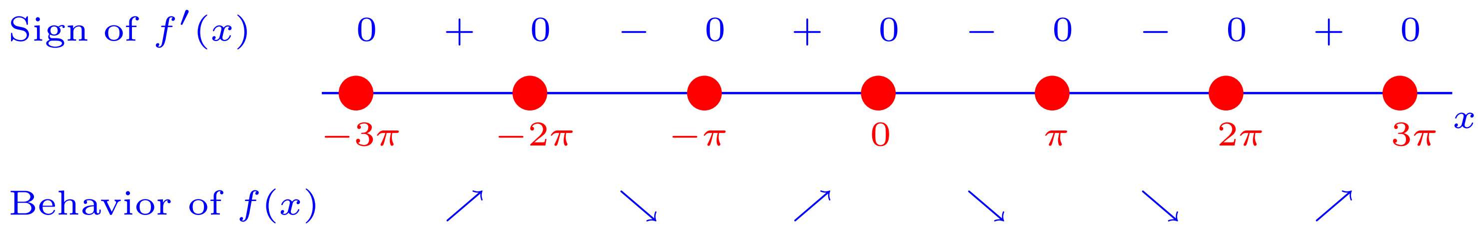

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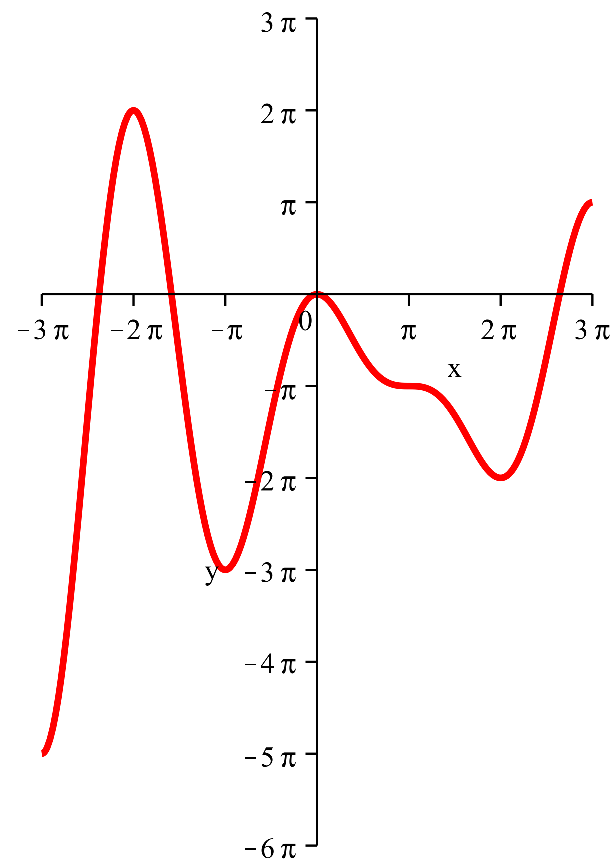

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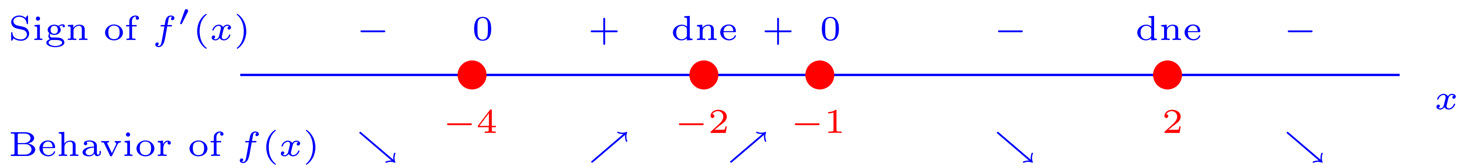

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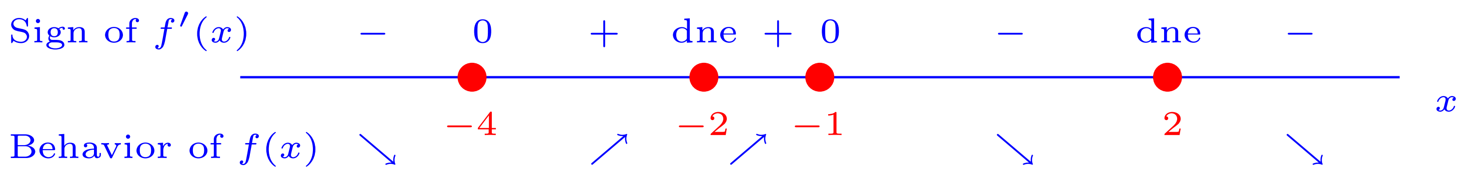

$f(x)$ decreases on $(-\infty, -4]$,
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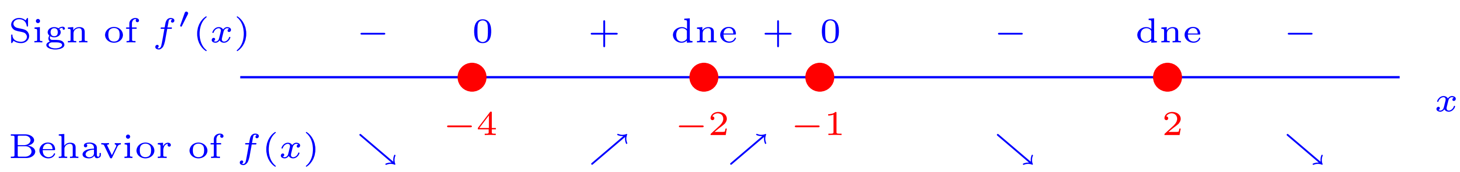

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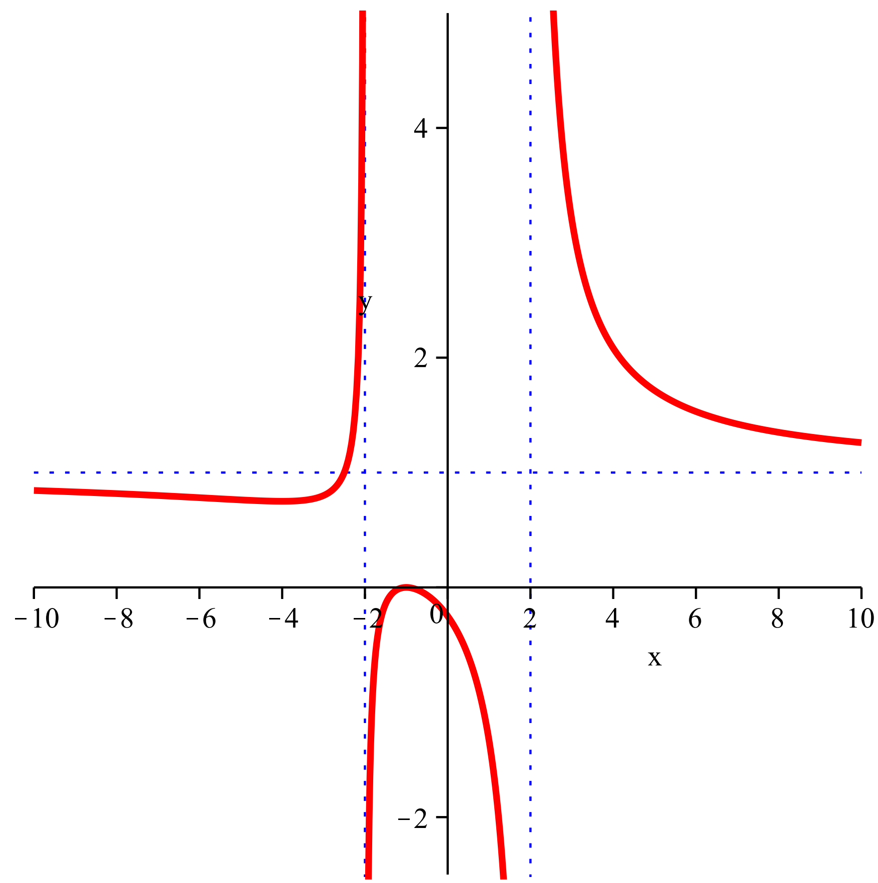

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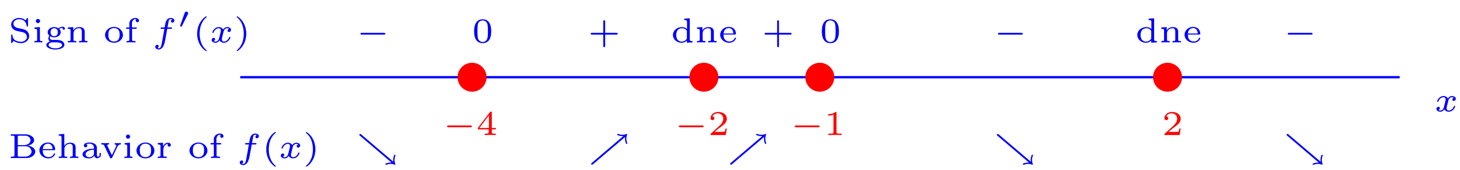

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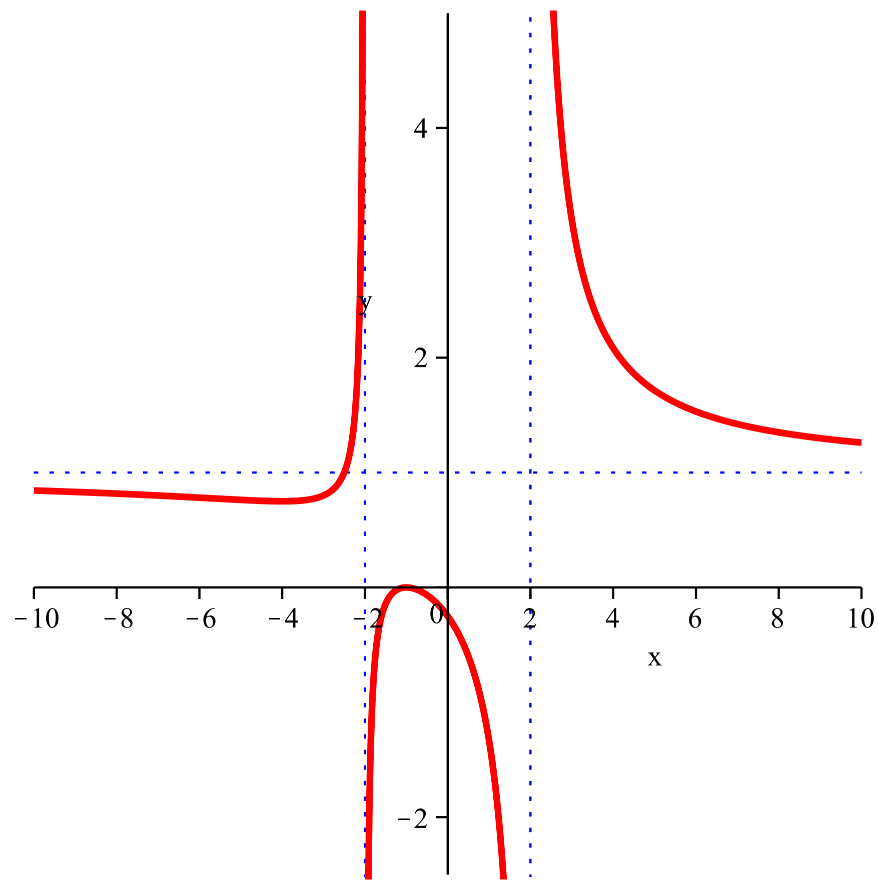

▶ $f(x)$ is NOT increasing
on $(-4,-2) \cup (-2,-1)$,
although $f' > 0$ on $(-4,-2) \cup (-2,-1)$.

Example 5. $f(x) = \begin{cases} x^3 - 6x + 1 & x \le 1, \\ x^2 - 6x + 7 & x > 1. \end{cases}$

$$f'(x) = \begin{cases} 3x^2 - 6 & x < 1, \\ 2x - 6 & x > 1. \end{cases}$$

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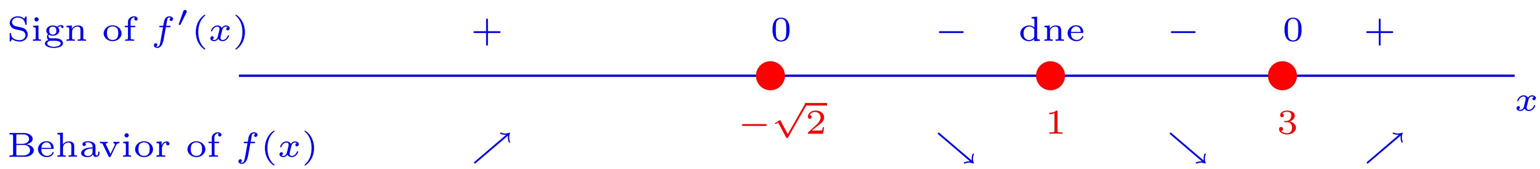

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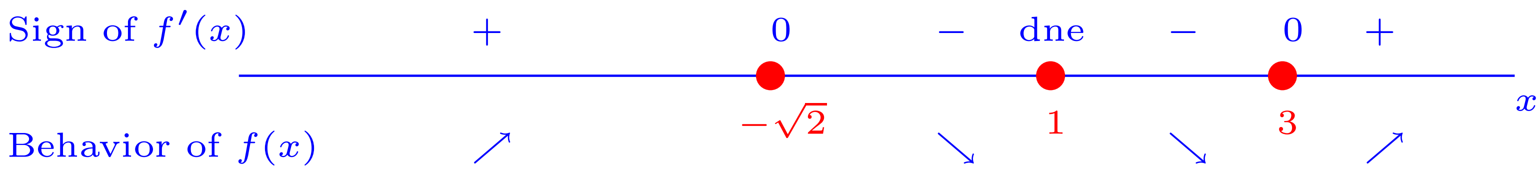

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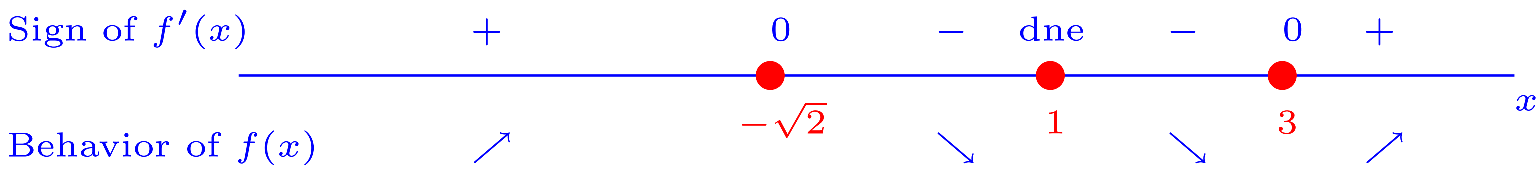

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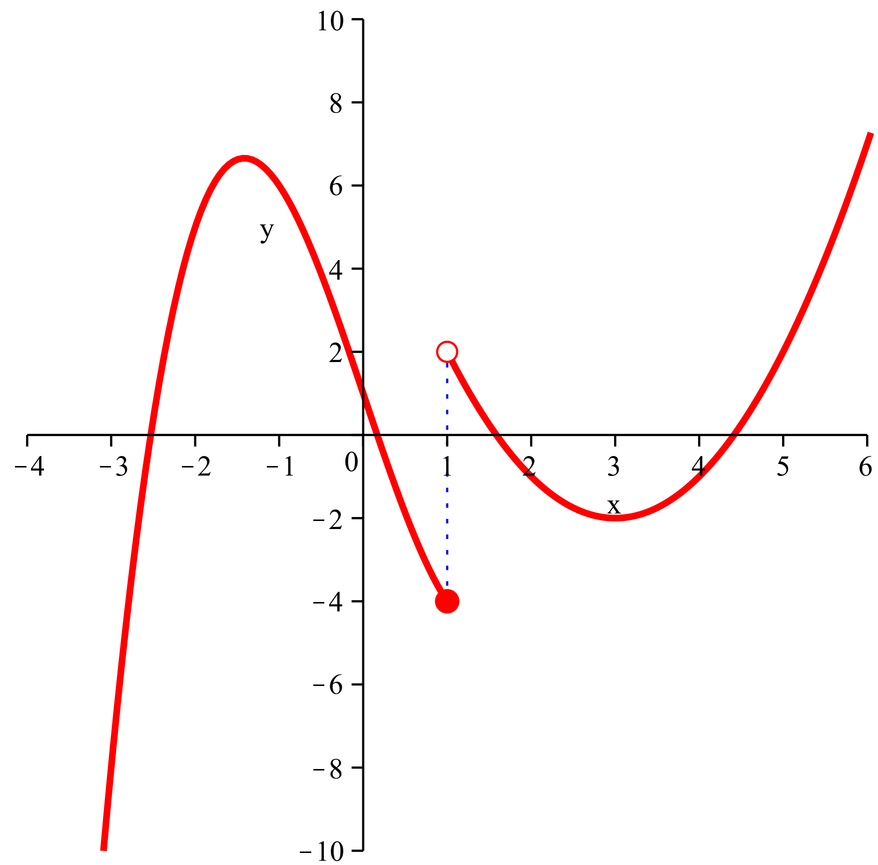

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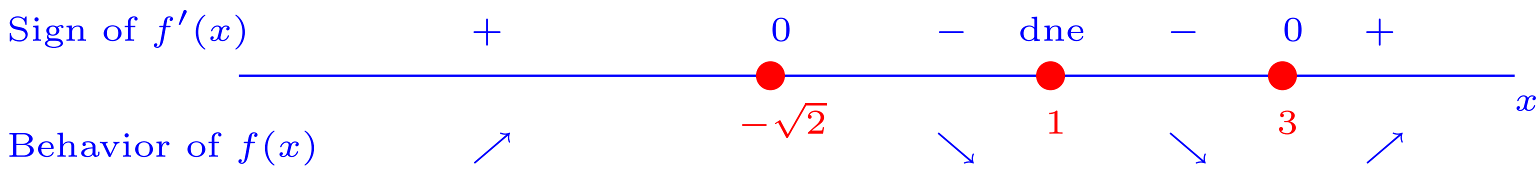

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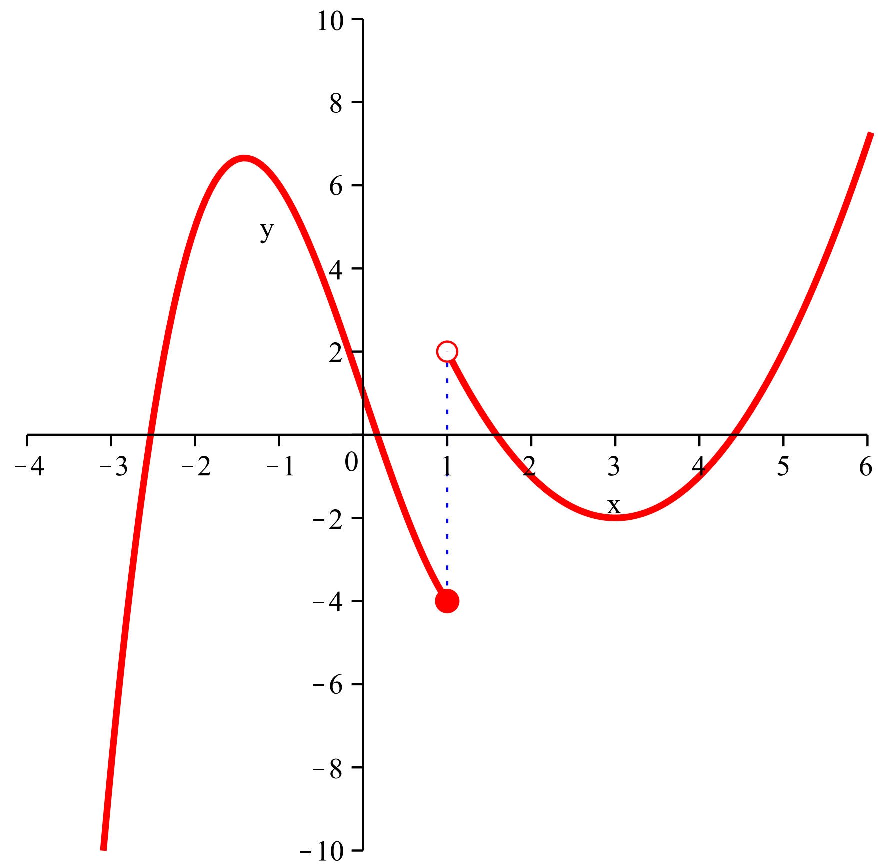

► $f(x)$ is NOT decreasing on $(-\sqrt{2}, 3)$, although $f' < 0$ on $(-\sqrt{2}, 1) \cup (1, 3)$.

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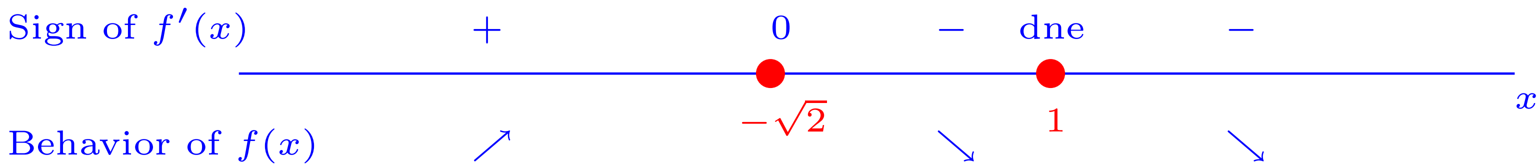

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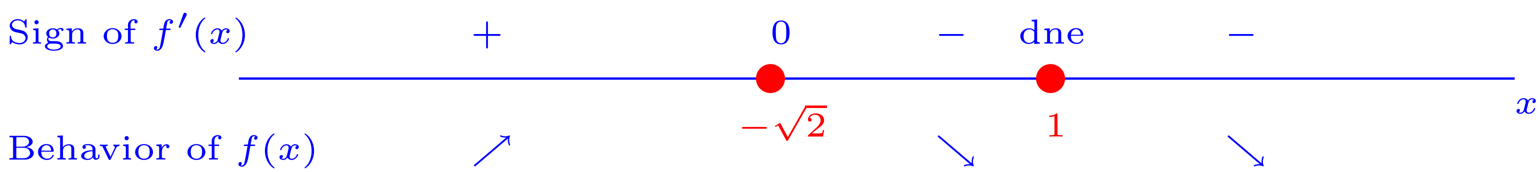

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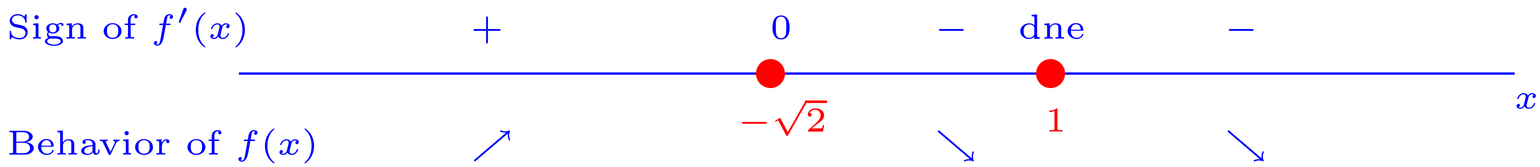

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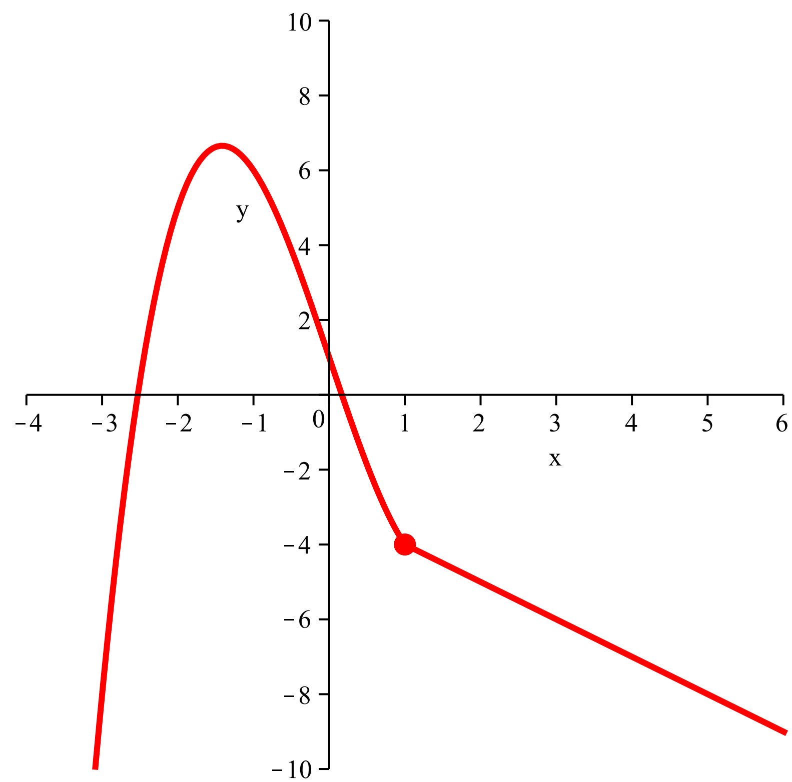

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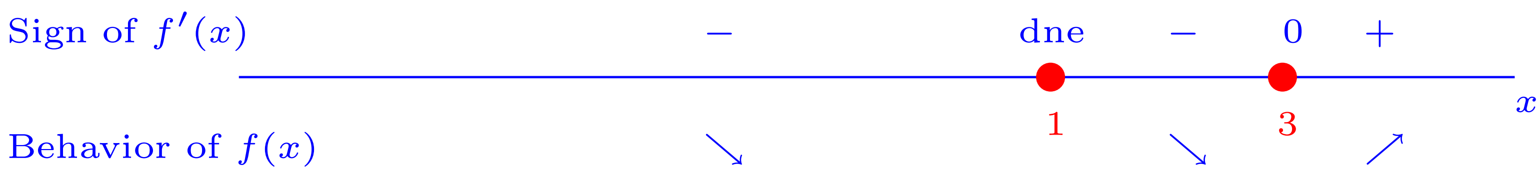

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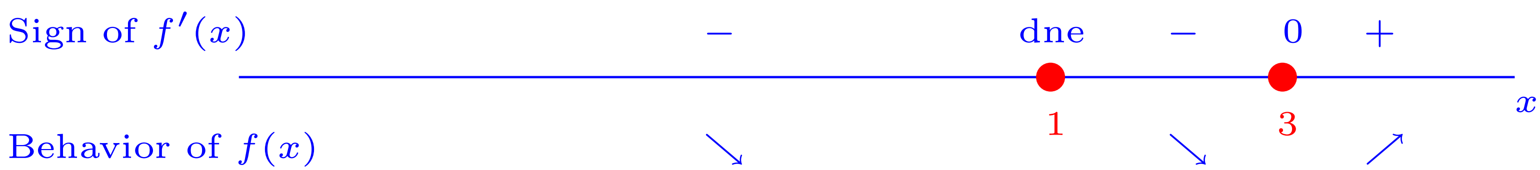

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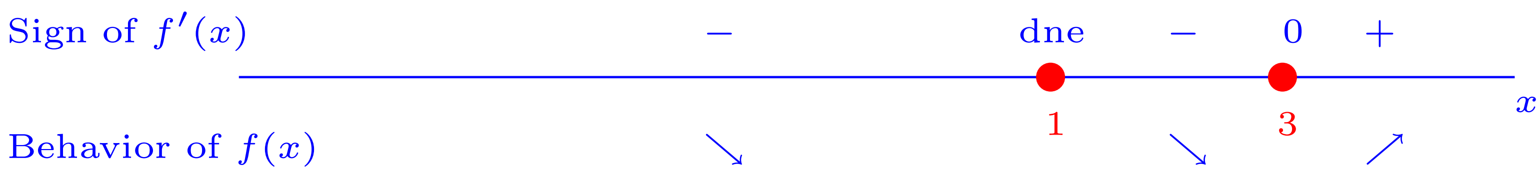

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